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MATHEMATICAL REPRESENTATIONS OF TURBULENT MIXING

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FOREWORD

This document is one of two documents that constitute the final report for Contract NAS8-28089, "Study of Viscous Mixing Plume Flow Field." This study was performed by the Lockheed-Huntsville Research & Engineering Center, Inc., for the National Aeronautics & Space Administration, George C. Marshall Space Flight Center, Alabama.

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CONTENTS

NOMENCLATURE

Section 2

A	scalar
$\overline{A}$	vector
$\overline{\overline{A}}$	tensor
A N ij	component of the tensor $\overline{\overline{A}}$ in the coordinate directions $\overline{W}$ Ni, $\overline{W}$ N j
A P i j	component of the tensor $\overline{\overline{A}}$ in the coordinate directions $\overline{P}$i, $\overline{P}$ j
A X i	component of the vector $\overline{A}$ in the coordinate direction $\overline{U}$ i
A Y i	component of the vector $\overline{A}$ in the coordinate direction $\overline{V}$ i
A Z i j	component of the tensor $\overline{\overline{A}}$ in the coordinate directions $\overline{W}$ i, $\overline{W}$ j
A Ø i j	component of the tensor $\overline{\overline{A}}$ in the coordinate directions $\overline{W}$ Ø i, $\overline{W}$ Ø j
B i j	matrix function of $\overline{\overline{G}}$
$\overline{\overline{B}}$	second-order tensor
$\overline{\overline{B}}$ A	antisymmetric, second-order tensor
$\overline{\overline{B}}$ S	symmetric, second-order tensor
C, C S	direction cosines
D A . . .	components of $\overline{\overline{D}}$ in the $\overline{W}$ A coordinate directions
$\overline{\overline{D}}$	tensor (Nth order)
E, ES	eigenvalue

EBij (EAij)	components of $\bar{\bar{E}}$E in the $\bar{W}$Bi, $\bar{W}$Bj ($\bar{W}$Ai, $\bar{W}$Aj) coordinate directions
$\bar{\bar{E}}$E	tensor (second-order)
$\bar{e}_i$	indicial symbol for $\bar{W}$Ni
$\bar{e}^i$	indicial symbol for $\bar{W}$Øi
FBi (FAi)	components of $\bar{F}$F in the $\bar{W}$Bi ($\bar{W}$Ai) coordinate direction
$\bar{F}$	vector of an antisymmetric tensor
$\bar{F}$F	vector
GCF	cofactors of $\bar{\bar{G}}$
$\bar{\bar{G}}$	general-coordinate stretching function, also called the metric tensor
$\bar{\bar{G}}$A, $\bar{\bar{G}}$B	general-coordinate metric tensors
$\bar{\bar{G}}$I or $\bar{\bar{G}}^{-1}$	inverse of $\bar{\bar{G}}$
g_{ij}	indicial symbol for Gij
g^{ij}	indicial symbol for GIij
$\bar{H}$, $\overline{HS}$	eigenvectors
$\bar{\bar{I}}$	unit vector
M	transformation matrix
MN	transformation matrix
N	transformation matrix
$\bar{P}$	general coordinate base vector
$\bar{R}$	position vector
$\bar{\bar{T}}$	second-order tensor
$\bar{U}$	Cartesian-coordinate unit vector
$\bar{V}$	orthogonal-coordinate unit vector
$\bar{W}$	general-coordinate unit vector

$\overline{W}A$, $\overline{W}B$	general-coordinate base vectors
$\overline{W}N$	general, contravariant-coordinate base vector
$\overline{W}\phi$	general, covariant-coordinate base vector
X	Cartesian-coordinate coordinate distance
X^i	indicial symbol for Z i
Y	orthogonal-coordinate coordinate distance
Z	general-coordinate coordinate distance
ZA, ZB	general-coordinate coordinate distances
ZN	general, contravariant-coordinate coordinate distance
$Z\phi$	general, covariant-coordinate coordinate distance
$\overline{\overline{\epsilon}}$	alternating tensor (third-order) (see Eq. 2.57)
θ	angle between $\overline{R}$ and $\overline{A}$

* * * * * * * * * * * * *

Lower case letters identify particular components of scalars, vectors and tensors

* * * * * * * * * * * * *

<u>Section 3</u>

A, B, C	constants
B	Nth moment
Ci	any dependent variable on pages 3-12, 3-13; constants elsewhere
$\overline{\overline{D}}$	rate-of-strain tensor
E	decay rate of turbulent energy
_F	any variable in the blank, and it becomes the fluctuating component of that variable
$\overline{\overline{G}}I$	inverse of the metric $\overline{\overline{G}}$, (see Section 2)
$\overline{\overline{I}}$	unit tensor

K	turbulent kinetic energy per unit mass
L, Li, Lij	length scales
_M	any variable in the blank, and it becomes the mean value of that variable
_ _N	components of the variable named in blanks in $\overline{W}$N coordinates
_ _$\emptyset$	components of the variable named in the blanks in the $\overline{W}\emptyset$ coordinates
P	pressure
$\overline{R}$	position vector
T	temperature
$\overline{U}$	Cartesian-coordinate base vector
$\overline{V}$	velocity
$\overline{W}$	vorticity vector
X	Cartesian-coordinate coordinate distance
XG2	lateral distance between two points on the velocity profile
β	second coefficient of viscosity
$\overline{\overline{\delta}}$	Kronecker delta (see Eq. (3.13))
ϵ	eddy viscosity
$\overline{\overline{\overline{\epsilon}}}$	alternating tensor (third-order) (see Eq. (2.57))
θ	time
μ	viscosity
ν	kinematic viscosity
ρ	density
$\overline{\overline{\sigma}}$	pressure tensor
$\overline{\overline{\tau}}$	shear-stress tensor

Section 4

A	damping factor
C, C_1	constants
L	length scale
ℓ	Van Driest mixing length
N	correlation parameter
P	pressure
Δtm	thickness of the mixing region
U	velocity along mixing region
V	velocity normal to mixing region
Y	lateral coordinate
δ	boundary
ϵ	eddy viscosity
μ	molecular viscosity
ρ	density
τ	shear stress

Subscripts

℄	centerline values
e, ∞	values in an undisturbed flow
i	inside
o	outside

GENERAL

$\{ \}$	denotes functionality
$\langle \rangle$	denotes time average
$\bar{\ }$	denotes vectors

$=$	denotes tensors
$\lvert\ \rvert$	absolute values
$\lVert\ \rVert$	the determinant

Section 1
INTRODUCTION AND SUMMARY

Much has been written concerning what is not known about turbulent mixing. The important thing is that some things are known. Some experiments have been performed and some calculation procedures have been developed. Incompressible, parallel stream turbulent mixing has been experimentally studied and can be accurately mathematically modeled. Afterburning within a rocket plume and an injector element's combustion efficiency have not yet yeilded to experimental technology or to analytical endeavor. The turbulence literature of free shear layers lies between these two extremes – mostly on the side of the first example.

Turbulent kinetic energy models are now a fashionable representation of free shear layers. Their use does offer the potential for a significant advance in our understanding of turbulence, but this potential has not yet been exploited. Furthermore, much of the importance of these models lies in their tensor behavior. The mathematical foundations of such behavior is not well documented anywhere. Studies to date have been made only on chemically simple flows.

Much maligned eddy viscosity models are still the only ones which can be used on combusting flows. Even these models have not been developed to their full extent. For example, empirically determined lateral and streamwise variations could be used in an eddy viscosity formulation. (These variations would be similar in form to intermittency factors used for boundary layers.) However, actual variations have not been satisfactorily determined either by experiments or turbulent kinetic methods.

The purpose of this report is to give a basic description of the mathematical tools and models which are presently used to represent turbulent, free shear layers. In addition, several recommendations are made for ways in which current modeling techniques can be improved.

Section 2
MATHEMATICAL PRELIMINARIES

Before the laws which govern turbulent flows can be fully applied, one must have a proper appreciation of tensor calculus, certain integral theorems and averaging procedures, as well as an understanding of ordinary and partial differential equations. The purpose of using tensor nomenclature is to ensure that complex terms are properly modeled and to simplify coordinate transformations; it also serves as a type of shorthand, but this is a trivial consequence. Integral theorems are needed if several control volumes are to be related. Averaging procedures are used to convert microscopic fluid properties to macroscopic ones.

This section presents the mathematical tools which are necessary to perform fundamental flow analyses.

2.1 SPACES AND FIELDS

In fluid mechanics, one is interested in predicting the fluid properties density, pressure, temperature, and composition, and the flow properties of velocity and head as a function of position and time. Unfortunately, investigators of other disciplines of interest to the fluid mechanicist, such as numerical analysis and statistical mechanics, have borrowed words to describe phenomena of interest to themselves. Therefore terms like vectors, tensors, coordinates, spaces, may have several meanings which are only loosely connected.

Historically, geometry and algebra problems were solved before some of the more complicated problems in functional analysis; hence, analogies were made and terms "generalized." Since this occurrence has now been perpetuated for several score years, the current mathematical literature has become very lax in separating the real from the abstract. An individual's unguided initial readings in these subjects are bewildering because a proper appreciation of the use of supposedly well-defined technical terms is lacking. This section attempts to provide a reading guide for this subject.

First, a field is defined as a scalar, vector or tensor-dependent variable which is evaluated over a region of space (call this "position" so "space" can be given a different meaning later) and time. Clearly, position and time are independent variables. Position is defined by three scalar components measured in three coordinate directions. The coordinates need not be rectilinear nor orthogonal. Certainly, the distance between two positions is defined; this distance is said to be measurable by a "metric."

The ideas of fields have been generalized by analogy into a description of a "function space" which may have any number of coordinates and which may relate all, some, or none of these coordinates by a metric.

To make these definitions more meaningful, let us say that three dialects of vector and tensor calculus could be "spoken;" namely, those that apply to a field which is completely specified by geometry, to a field which is specified by geometry and time, and to a field which is specified by any number of coordinates. These fields will be named: Geometric Fields, Unsteady Fields, and Function Spaces, respectively. In Function Space the coordinates may or may not be related by a metric, and furthermore, the metric may or may not have geometric meaning.

The language used to describe each of these fields is defined in following paragraphs and then arranged in a form such that it may, subsequently, be used.

2.2 GEOMETRIC FIELDS

Physical laws must be independent of the units chosen to represent physical quantities; hence, units in equations are carefully accounted for. Physical laws must also be independent of the coordinates. There are three types of geometric coordinate systems: orthogonal rectilinear, orthogonal

B

curvilinear, and general curvilinear. In each of these systems a specific set of rules is used to describe the relationship between flow properties and geometrical position. These coordinate systems and these rules are described in this section.

Certain types of quantities which appear in physical laws must be classified, because they transform from one coordinate system to another by different rules. These quantities are defined and illustrated with examples from fluid mechanics. The first of these is a scalar quantity, which requires that magnitude only be specified. Examples are: mass, density, energy, temperature, volume, pressure, time. The second of these is a vector quantity which requires that a magnitude and a direction be specified. In fluid mechanics, three scalar values associated with convenient orthogonal directions are usually employed to specify a vector. Examples of vectors are: velocity, momentum, acceleration and force.

There are more complicated quantities which require that a larger number of scalar quantities be specified. These quantities are called tensors, a very general term which may be applied to quantities of any degree of complexity. Thus, a tensor of the "zero-order" ("zero-rank") means that 3^0 scalars must be specified — this is our "scalar" for a three-dimensional space. A tensor of the first-order means that 3^1 scalars must be specified as our "vectors." A second-order tensor requires that 3^2, nine scalars be specified. A third-order tensor requires that 27 scalar components be specified. This is the highest order tensor usually necessary in fluid mechanics.

A second-order tensor requires that a magnitude and two directions be specified. In describing the relationship between force, area, and stress, the force-vector was described by a magnitude and direction. To describe stress, the direction between the area over which the stress acts as well as the force must be specified. Examples of second-order tensors are: stress, rate-of-strain, momentum-flux, mass moment of inertia.

A geometric field is a distribution of scalar, vector, or tensor quantities described by functions of space coordinates. The nomenclature for making such a description is now to be established.

In general, upper case letters denote variable names. Sometimes more than one letter is used to name a variable. Lower case letters and numbers indicate particular components of a variable. Indicial notation was avoided because it is difficult to type and to write on a blackboard. The capital letter system introduced herein is designed to provide a minimum of symbol changes when equations are solved with a FORTRAN program.

2.3 COORDINATE SYSTEMS

Coordinate systems are so basic to the understanding of engineering problems that their existence and properties are usually presumed to be known to the reader. The next few pages attempt to document this "common" knowledge. The nomenclature that is needed for subsequent discussion is also defined. From the onset, a distinction is made between the coordinate systems used in engineering analysis and those which have more application to mathematical analysis. The understanding of three engineering and two mathematical coordinate systems is essential.

2.3.1 Engineering Coordinates

An engineering coordinate system is specified by defining:

1. Unit vectors,
2. Coordinate names,
3. Stretching functions.

Unit vectors and coordinates are named with letters and a number; one number for each of the three directions. The three common coordinate systems are:

Cartesian, denoted by $\overline{U}$, X as the unit vector and coordinate, respectively;

Orthogonal curvilinear, denoted by $\overline{V}$, Y,

General curvilinear, denoted by $\overline{W}$, Z.

All of the coordinate systems used herein are taken to have the same origin. A point in space is located by a position vector $\overline{R}$. In a Cartesian system,

$$\overline{R} = \overline{U}1 * X1 + \overline{U}2 * X2 + \overline{U}3 * X3 \tag{2.1}$$

In any other type coordinate system, the definition of $\overline{R}$ becomes somewhat abstract because the unit vectors vary between the origin and the point; therefore, basic definitions are established for an incremental departure from a point, $d\overline{R}$. To accomplish this definition, stretching functions which are the relationships between increments of coordinates and distances tangent to the coordinate directions must be introduced. If a coordinate is an angle, the stretching function converts the increment of angle into an increment of length. Now

$$\left.\begin{aligned} d\overline{R} &= \overline{U}1 * dX1 + \overline{U}2 * dX2 + \overline{U}3 * dX3 \\ &= \overline{V}1 * (H1)^{1/2} * dY1 + \overline{Y}2 * (H2)^{1/2} * dY2 + \overline{Y}3 * (H3)^{1/2} * dY3 \\ &= \overline{W}1 * (G11)^{1/2} * dZ1 + \overline{W}2 * (G22)^{1/2} * dZ2 + \overline{W}3 * (G33)^{1/2} * dZ3 \end{aligned}\right\} \tag{2.2}$$

for each of the coordinate systems, where the G's and the H's represent the stretching functions. The reason for using the numbers in these terms is explained subsequently. Note that stretching functions for the Cartesian system are unity.

The values of the unknown stretching functions are determined by first calculating ($d\overline{R} \cdot d\overline{R}$) and then formally transforming from the Cartesian system to the curvilinear system of interest.

Recalling that the dot product of two vectors is the product of their magnitudes times the cosine of the angle between them,

$$
\begin{aligned}
(d\overline{R} \cdot d\overline{R}) &= (dR)^2 \\
&= (dX1)^2 + (dX2)^2 + (dX3)^2 \\
&= H1\ (dY1)^2 + H2\ (dY2)^2 + H3\ (dY3)^2 \\
&= G11\ (dZ1)^2 + \cos(1,2)\ (G11)^{1/2}\ (G22)^{1/2}\ (dZ1)\ (dZ2) + \\
&\quad \cos(1,3)\ (G11)^{1/2}\ (G33)^{1/2}\ (dZ1)\ (dZ3) + \\
&\quad \cos(2,1)\ (G22)^{1/2}\ (G11)^{1/2}\ (dZ2)\ (dZ1) + (G22)(dZ2)^2 + \\
&\quad \cos(2,3)\ (G22)^{1/2}\ (G33)^{1/2}\ (dZ2)\ (dZ3) + \\
&\quad \cos(3,1)\ (G33)^{1/2}\ (G11)^{1/2}\ (dZ3)\ (dZ1) + \\
&\quad \cos(3,2)\ (G33)^{1/2}\ (G22)^{1/2}\ (dZ3)\ (dZ2) + (G33)(dZ3)^2
\end{aligned}
\tag{2.3}
$$

where cos (i, j) means the cosine of the angle between $\overline{W}i$ and $\overline{W}j$.

Suppose that the Cartesian coordinates are expressed in terms of new coordinates by the equations:

$$
\begin{aligned}
X1 &= X1\ \{Z1, Z2, Z3\} \\
X2 &= X2\ \{Z1, Z2, Z3\} \\
X3 &= X3\ \{Z1, Z2, Z3\}
\end{aligned}
\tag{2.4}
$$

where the braces denote that Xi is a function of Z1, Z2, Z3. The unit vectors and stretching functions are not to be considered as independent variables.

The formal transformation of coordinates is

$$dXi = \sum Mmi\ dZm \tag{2.5}$$

where

$$
Mmi = \frac{\partial Xi}{\partial Zm} = \begin{bmatrix} \frac{\partial X1}{\partial Z1} & \frac{\partial X1}{\partial Z2} & \frac{\partial X1}{\partial Z3} \\ \frac{\partial X2}{\partial Z1} & \frac{\partial X2}{\partial Z2} & \frac{\partial X2}{\partial Z3} \\ \frac{\partial X3}{\partial Z1} & \frac{\partial X3}{\partial Z2} & \frac{\partial X3}{\partial Z3} \end{bmatrix} \tag{2.6}
$$

and the summation is performed on the repeated symbol, m. (Note: neither unit vectors nor stretching functions have yet been specified in our transformation scheme. Note also, that the determinant of the matrix M is the Jacobian of the transformation.)

Now the incremental distance can be determined by

$$
(dR)^2 = \sum\sum\sum (Mmi)\,(dZm)\,(Mni)\,(dZn) \tag{2.7}
$$

where the sums are on m, n, i over 1,2,3. It must be remembered that the "Mmi" terms in Eqs. (2.5) and (2.7) are matrices, furthermore since the dZ terms in Eq. (2.7) do not contain "i", the M's may be summed first. This summation of M's is called the "metric," Gmn. It is a matrix; other properties of significance that it has are discussed later.

These operations are summarized in Fig. 1.

By comparing terms in the two length expressions Eqs. (2.3) and (2.7) observe:

$$
Gki = Gik = (Gii)^{1/2}\,(Gkk)^{1/2}\cos(i, k)
$$

therefore the matrix Gmn is symmetric. The significance of these symmetry properties is given in the comments section at the end of this section.

a. The Metric – $\sum$ (Mmi) (Mni) = Gmn =

$$
\begin{matrix}
\left[\frac{\partial X1}{\partial Z1}\frac{\partial X1}{\partial Z1}+\frac{\partial X2}{\partial Z1}\frac{\partial X2}{\partial Z1}+\frac{\partial X3}{\partial Z1}\frac{\partial X3}{\partial Z1}\right] & \left[\frac{\partial X1}{\partial Z1}\frac{\partial X1}{\partial Z2}+\frac{\partial X2}{\partial Z1}\frac{\partial X2}{\partial Z2}+\frac{\partial X3}{\partial Z1}\frac{\partial X3}{\partial Z2}\right] & \left[\frac{\partial X1}{\partial Z1}\frac{\partial X1}{\partial Z3}+\frac{\partial X2}{\partial Z1}\frac{\partial X2}{\partial Z3}+\frac{\partial X3}{\partial Z1}\frac{\partial X3}{\partial Z3}\right] \\
\left[\frac{\partial X1}{\partial Z2}\frac{\partial X1}{\partial Z1}+\frac{\partial X2}{\partial Z2}\frac{\partial X2}{\partial Z1}+\frac{\partial X3}{\partial Z2}\frac{\partial X3}{\partial Z1}\right] & \left[\frac{\partial X1}{\partial Z2}\frac{\partial X1}{\partial Z2}+\frac{\partial X2}{\partial Z2}\frac{\partial X2}{\partial Z2}+\frac{\partial X3}{\partial Z2}\frac{\partial X3}{\partial Z2}\right] & \left[\frac{\partial X1}{\partial Z2}\frac{\partial X1}{\partial Z3}+\frac{\partial X2}{\partial Z2}\frac{\partial X2}{\partial Z3}+\frac{\partial X3}{\partial Z2}\frac{\partial X3}{\partial X3}\right] \\
\left[\frac{\partial X1}{\partial Z3}\frac{\partial X1}{\partial Z1}+\frac{\partial X2}{\partial Z3}\frac{\partial X2}{\partial Z1}+\frac{\partial X3}{\partial Z3}\frac{\partial X3}{\partial Z1}\right] & \left[\frac{\partial X1}{\partial Z3}\frac{\partial X1}{\partial Z2}+\frac{\partial X2}{\partial Z3}\frac{\partial X2}{\partial Z2}+\frac{\partial X3}{\partial Z3}\frac{\partial X3}{\partial Z2}\right] & \left[\frac{\partial X1}{\partial Z3}\frac{\partial X1}{\partial Z3}+\frac{\partial X2}{\partial Z3}\frac{\partial X2}{\partial Z3}+\frac{\partial X3}{\partial Z3}\frac{\partial X3}{\partial Z3}\right]
\end{matrix}
$$

$$
= \begin{pmatrix} G11 & G12 & G13 \\ G21 & G22 & G23 \\ G31 & G32 & G33 \end{pmatrix} \qquad (2.8)
$$

Note: Gnm = Gmn; i.e., matrix is symmetric.
The summation is on i.

b. Distance –

$$
\begin{aligned}
(dR)^2 = {} & (G11)\,(dZ1)\,(dZ1) + (G12)\,(dZ1)\,(dZ2) + (G13)\,(dZ1)\,(dZ3) + \\
& (G21)\,(dZ2)\,(dZ1) + (G22)\,(dZ2)\,(dZ2) + (G23)\,(dZ2)\,(dZ3) + \\
& (G31)\,(dZ3)\,(dZ1) + (G32)\,(dZ3)\,(dZ2) + (G33)\,(dZ3)\,(dZ3)
\end{aligned} \qquad (2.9)
$$

c. $\|Gnm\| = G11\,G22\,G33 + G12\,G23\,G31 + G13\,G21\,G32 - G13\,G22\,G31 - G11\,G23\,G32 - G12\,G21\,G33$

$$
= \frac{\partial(X1, X2, X3)}{\partial(Z1, Z2, Z3)}, \text{ the Jacobian} \qquad (2.10)
$$

Fig. 1 - Properties of the Metric

To complete the transformation, unit vectors in the curvilinear coordinate system must be determined. Eq. (2.2) may be written as

$$d\overline{R} = \frac{\partial \overline{R}}{\partial Z1} dZ1 + \frac{\partial \overline{R}}{\partial Z2} dZ2 + \frac{\partial \overline{R}}{\partial Z3} dZ3 \tag{2.11}$$

but

$$\overline{R} = \overline{U}1\ X1 + \overline{U}2\ X2 + \overline{U}3\ X3 \tag{2.12}$$

Therefore,

$$\overline{W}1\,(G11)^{1/2} = \frac{\partial X1}{\partial Z1}\overline{U}1 + \frac{\partial X2}{\partial Z1}\overline{U}2 + \frac{\partial X3}{\partial Z1}\overline{U}3 \tag{2.13}$$

$$\overline{W}2\,(G22)^{1/2} = \frac{\partial X1}{\partial Z2}\overline{U}1 + \frac{\partial X2}{\partial Z2}\overline{U}2 + \frac{\partial X3}{\partial Z2}\overline{U}3 \tag{2.14}$$

$$\overline{W}3\,(G33)^{1/2} = \frac{\partial X1}{\partial Z3}\overline{U}1 + \frac{\partial X2}{\partial Z3}\overline{U}2 + \frac{\partial X3}{\partial Z3}\overline{U}3 \tag{2.15}$$

The orthogonal curvilinear system is a special case of the general curvilinear system which is developed in these pages. Its development would show

$$Hi \text{ to equal } Gii$$

and

$$Gij = 0 \quad \text{for} \quad i \neq j\,.$$

Transformations could have been made directly between any two coordinate systems, but it is convenient to consider the Cartesian system as the reference. (Be cautioned that all literature in this field does not use this reference.)

2.3.2 Mathematical Coordinates

There are several other coordinate systems which may be defined in terms of the metric Gmn. In general they are non-orthogonal, so they are named $\overline{W}$__, Z__ where additional letters are supplied to name each specific coordinate system. These mathematical coordinate systems use only base vectors and coordinate increments; i.e., the stretching functions are not defined explicitly but are part of the base vectors. Two additional systems are used: those tangential to coordinate lines — denoted by N in the second letter of their name and those normal to coordinate lines — denoted by $\emptyset$.

If base vectors are defined by

$$\overline{WNi} = (Gii)^{1/2}\,\overline{Wi} \tag{2.16}$$

they will be parallel to $\overline{Wi}$ (and also to the coordinate i) but will not be of unit magnitude. Coordinate lengths will be denoted by ZNi; hence, incremental distances become

$$d\overline{R} = \overline{WN1}\,dZN1 + \overline{WN2}\,dZN2 + \overline{WN3}\,dZN3 \tag{2.17}$$

Notice that dZNi = dZi and that stretching functions are not used. The base vector transformations to a Cartesian coordinate system are:

$$\overline{WNj} = \sum \frac{\partial Xi}{\partial ZNj}\,\overline{Ui} \qquad \text{sum } i = 1, 2, 3; \text{ to get } j \text{ eqs.} \tag{2.18}$$

Hence the $\overline{WN}$ base vectors can be obtained from the $\overline{U}$ unit vectors by a linear-differential-transformation by using the matrix:

$$Nkm \equiv \frac{\partial Xm}{\partial ZNk} = \begin{pmatrix} \frac{\partial X1}{\partial ZN1} & \frac{\partial X2}{\partial ZN1} & \frac{\partial X3}{\partial ZN1} \\ \frac{\partial X1}{\partial ZN2} & \frac{\partial X2}{\partial ZN2} & \frac{\partial X3}{\partial ZN2} \\ \frac{\partial X1}{\partial ZN3} & \frac{\partial X2}{\partial ZN3} & \frac{\partial X3}{\partial ZN3} \end{pmatrix} \tag{2.19}$$

Notice that $\overline{WNi} \cdot \overline{WNj} = Gij$.

Also

$$(dR)^2 = \sum\sum\sum (MNmi\ dZNm)(MNni\ dZNn) \quad \text{sum } m, n, i = 1, 2, 3$$

and

$$\sum MNmi\ MNni = Gmn \quad \text{sum } i = 1, 2, 3.$$

To further confound the reader, another set of base vectors which are perpendicular to the coordinate surfaces, as opposed to being parallel to the coordinate directions, is to be defined. Xi Cartesian coordinates and Yi orthogonal coordinates are both parallel to the i coordinate direction and perpendicular to the j and k coordinate surfaces. Only when non-orthogonal coordinates are used is there a difference in these directions.

The new system is in terms of $\overline{W}\phi$, $Z\phi$. The direction requirements for this perpendicular coordinate system are:

$$\begin{aligned} WN1 \cdot W\phi 2 &= 0 \\ WN1 \cdot W\phi 3 &= 0 \\ WN2 \cdot W\phi 3 &= 0 \\ WN2 \cdot W\phi 1 &= 0 \\ WN3 \cdot W\phi 1 &= 0 \\ WN3 \cdot W\phi 2 &= 0 \end{aligned} \tag{2.20}$$

Let

$$d\overline{R} = \frac{\partial \overline{R}}{\partial Z\phi 1} * dZ\phi 1 + \frac{\partial \overline{R}}{\partial Z\phi 2} * dZ\phi 2 + \frac{\partial \overline{R}}{\partial Z\phi 3} * dZ\phi 3 \tag{2.21}$$

and

$$dXi = \sum \frac{\partial Xi}{\partial Z\phi j} dZ\phi j \quad \text{sum } j = 1, 2, 3; \text{ to get } i \text{ eqs.} \tag{2.22}$$

where

$$Xi = Xi \{Z\phi 1,\ Z\phi 2,\ Z\phi 3\} \text{ for } i = 1, 2, 3$$

Therefore a matrix like Eq. (2.6) and a metric like Eq. (2.8) can be constructed. If base vectors $\overline{W\phi i}$ are defined by

$$\overline{W}\phi i = \frac{\partial \overline{R}}{\partial Z\phi i} =$$

$$\sum \frac{\partial Xj}{\partial Z\phi i} \quad \overline{Uj} \quad \text{Sum } j = 1, 2, 3; \text{ to get } i \text{ eqs.} \tag{2.23}$$

Now how are the functions chosen so that the $\overline{W}\phi i$'s will be normal to coordinate surfaces?

In determining the $\overline{WN}$, ZN coordinate system, it was sufficient to evaluate the relative partial derivatives of the coordinate lines or the elements of the metric. Therefore, let us consider other properties of this matrix. Other more direct methods could be attempted (see comment). The inverse of $\overline{\overline{G}}$ is: $\overline{\overline{G}}^{-1}$. $\overline{\overline{G}}$ is given a tensor symbol; proof of tensor properties is given in Ref. 2-1, p. 107. It is evaluated thusly.

1. Given

$$\overline{\overline{G}} = \begin{pmatrix} G11 & G12 & G13 \\ G21 & G22 & G23 \\ G31 & G32 & G33 \end{pmatrix} \tag{2.24}$$

2. The determinate of $\overline{\overline{G}}$ is $\|Gnm\|$ and it is Eq. (2.10).

3. The cofactors of $\bar{\bar{G}}$ are (Ref. 2-1, p.49):

$$
\begin{aligned}
GCF11 &= (-1)^2 \,(G22\,G33 - G23\,G32) \\
GCF21 &= (-1)^3 \,(G12\,G33 - G32\,G13) \\
GCF31 &= (-1)^4 \,(G12\,G23 - G22\,G13) \\
GCF12 &= (-1)^3 \,(G21\,G33 - G31\,G23) \\
GCF22 &= (-1)^4 \,(G11\,G33 - G31\,G13) \\
GCF32 &= (-1)^5 \,(G11\,G23 - G21\,G13) \\
GCF13 &= (-1)^4 \,(G21\,G32 - G31\,G22) \\
GCF23 &= (-1)^5 \,(G11\,G32 - G31\,G12) \\
GCF33 &= (-1)^6 \,(G11\,G22 - G21\,G12)
\end{aligned}
\tag{2.25}
$$

4. Define

$$
\bar{\bar{G}}CF = \begin{pmatrix} GCF11 & CGF12 & GCF13 \\ GCF21 & GCF22 & GCF23 \\ GCF31 & GCF32 & GCF33 \end{pmatrix}
\tag{2.26}
$$

5. The transpose of

$$
\bar{\bar{\bar{G}}}CF = \begin{pmatrix} GCF11 & GCF21 & GCF31 \\ GCF12 & GCF22 & GCF32 \\ GCF13 & GCF23 & GCF33 \end{pmatrix}
\tag{2.27}
$$

6. Lastly,

$$
\bar{\bar{G}}^{-1} = \begin{pmatrix} \frac{GCF11}{\|Gnm\|} & \frac{GCF21}{\|Gnm\|} & \frac{GCF31}{\|Gnm\|} \\ \frac{GCF12}{\|Gnm\|} & \frac{GCF22}{\|Gnm\|} & \frac{GCF32}{\|Gnm\|} \\ \frac{GCF13}{\|Gnm\|} & \frac{GCF23}{\|Gnm\|} & \frac{GCF33}{\|Gnm\|} \end{pmatrix}
\tag{2.28}
$$

Let $\bar{\bar{G}}^{-1} = \bar{\bar{G}}I$ (2.29)

Now if $\bar{\bar{G}}I$ is chosen as the metric, then the $\overline{W\phi}$, $Z\phi$ coordinate system will be correctly aligned with the normals to the coordinate surfaces. This implies that the $\bar{U}$, X and $\overline{WN}$, ZN coordinate systems must both be considered before the $\overline{W\phi}$, $Z\phi$ can be defined; i.e., elements of GI are calculated from these other two systems. Hence;

$$\sum_i \frac{\partial Xi}{\partial Z\phi j} \; \frac{\partial Xi}{\partial Z\phi k} = GIjk \tag{2.30}$$

and

$$GIjk = \overline{W\phi}j \bullet \overline{W\phi}k \tag{2.31}$$

On occasion, mixtures of coordinate systems are used. For example, a $\overline{WN}1$, ZN1; $\overline{WN}2$, ZN2; $\overline{W\phi}3$, $Z\phi3$ system could be used. These possibilities are considered again on subsequent pages.

2.4 THE ORIGIN OF TENSORS

Discounting zero and first-order tensors which are completely described by vector analysis, when do tensor quantities arise in engineering analysis? Products of vectors without dot or cross simplifications and division by vectors are not defined in vector analysis. These quantities arise in the physical world and in the mathematical world. They are second (and higher) order tensors.

The product of two vectors is a second order tensor called a dyadic. Other quantities have also been given special names, but they are not reviewed in this report.

Now vectors and tensors are defined; and, for consistency, scalars are also named at this point.

Zero-order tensor (scalar): $A = A$, i.e., one term. (2.32)

First-order tensor (vector): $\overline{A} = (AZ1)(\overline{W}1) + (AZ2)(\overline{W}2) + AZ3(\overline{W}3)$ (2.33)

Notice there are three scalar coefficients, each of which is associated with a coordinate direction. AZ1, AZ2, AZ3 may represent the velocity components, for example.

Second-order tensor:
$$\begin{aligned}\overline{\overline{A}} = {} & (\overline{W}1)(AZ11)(\overline{W}1) + (\overline{W}2)(AZ22)(\overline{W}2) + (\overline{W}3)(AZ33)(\overline{W}3) \\ & + (\overline{W}1)(AZ12)(\overline{W}2) + (\overline{W}1)(AZ13)(\overline{W}3) + (\overline{W}2)(AZ21)(\overline{W}1) \\ & + (\overline{W}2)(AZ23)(\overline{W}3) + (\overline{W}3)(AZ31)(\overline{W}1) + (\overline{W}3)(AZ32)(\overline{W}2)\end{aligned} \quad (2.34)$$

Notice the one magnitude and two directions associated with each of the nine elements of this quantity. This may represent the stress relationships (AZij) between the force directions (the first $\overline{W}i$) and surface directions (the second $\overline{W}j$). Also, note the natural use of indices shown by lower case letters (i, j, k, etc.,) to represent general terms. The order of the terms in the triplet of products has no special significance; however, it must be remembered (or established by convention) which ($\overline{W}i$) is associated with force and which with surface (or any other pair of related quantities). Therefore the second-order tensor may be stated:

$$\overline{\overline{A}} = \sum\sum (AZij)(\overline{W}i)(\overline{W}j) \text{ for } i, j = 1, 2, 3 \quad (2.35)$$

Hence, an "N"th - order tensor is:

$$\overline{\overline{A}} = \sum\sum \cdots \sum (AZij \ldots m)(\overline{W}i)(\overline{W}j) \ldots (\overline{W}m) \quad \text{for } i, j, \ldots m = 1, 2, 3 \quad (2.36)$$

There are N summations, N unit vectors, and N indices on Aij...m. Obviously, for $N > 3$ some of the unit vectors must always appear more than one time; i.e., A1123 is a component for $N = 4$.

Tensors have now been defined. They have not been specifically defined by transformation laws, as they often are. Such transformation laws are discussed — in due time — and they do indeed hold true; but the definitions as stated are those in which the application of mathematical modeling of a physical situation is most obvious. The necessity of another defining mechanism is indicated if spaces rather than fields are of interest.

The literature on tensor calculus is not consistent in its use of terms. Perhaps a mathematician would rather say that he has generalized certain results. Nevertheless, both the $\bar{\bar{A}}$'s and the AZij's are called tensors. $\bar{\bar{A}}$ is independent of the selection of coordinates; i.e., it is invariant with respect to coordinates, whereas AZij's are not.

Returning to the definition of "N"th order tensor, and indicating the use of either $\overline{W}$i's or $\overline{W}$Ni's.

$$\bar{\bar{A}} = \sum\sum\cdots\sum (AZij\ldots m)(\overline{W}i)(\overline{W}j)\ldots(\overline{W}m) =$$

$$= \sum\sum\cdots\sum (ANij\ldots m)(\overline{W}Ni)(\overline{W}Nj)\ldots(\overline{W}Nm) \tag{2.37}$$

$$\text{for } i, j, \ldots m = 1, 2, 3$$

Note the corresponding tensor components AZij...m and ANij..m are not equal, although either, with the appropriate unit or base vectors, will suffice to define the invariant $\bar{\bar{A}}$. Unit vectors are also base vectors in orthogonal coordinate systems.

The ANij.. m are called contravariant components of the tensor $\bar{\bar{A}}$. If the $\overline{W}\phi$ vector had been used,

$$\bar{\bar{A}} = \sum\sum\cdot\cdot\sum (A\phi ij..m)(\overline{W}\phi i)(\overline{W}\phi j)..(\overline{W}\phi m) \tag{2.38}$$

The Aϕij..m would be called the covariant components of the tensor $\bar{\bar{A}}$.

Now to really complicate matters one might choose a mixture of base vectors. For example, consider a second order tensor.

$$\bar{\bar{A}} = \sum\sum (APij)(\bar{P}i)(\bar{P}j) \quad \text{for } i, j = 1, 2, 3 \tag{2.39}$$

where

$$\bar{P}i = \bar{W}Ni \text{ and } \bar{P}j = \bar{W}\phi j$$

Components in the $\bar{W}N$ coordinate system are named contravariant components and in $\bar{W}\phi$, they are called covariant. APij is said to be co-variant with respect to $\bar{W}\phi j$ and contravariant with respect to $\bar{W}Ni$'s. Often they are distinguished by using subscripts and superscripts. Since these components are related by the metric, Gmn, they can be interchanged at will.

The other major nomenclature system of interest is the indicial one. The indicial system strives to preserve the same symbol for the invariant and for its components; it distinguishes between the coordinate systems which can be defined by the metric Gmn by the location of the indices. Since the indicial system uses super- and subscripts for the distinctions between components, performing transformations is often referred to as "raising and lowering the indicies."

The tensor components which are used by the engineer are the AZij ones. The units of these components are the same in all coordinate systems. The other coordinate systems, hence the other components, do not have this property. Historically, in attempting to discuss the components of tensors, without first specifying the unit (or base) vectors which are being used, special names have been given to these components. AZij's are called physical tensors (Ref. 2-2, real tensors (Ref. 2-3) and true tensors (Ref. 2-4). In fact, Ref. 2-4 refers to other tensors as pseudo-tensors. Modern usage gives pseudo-tensors a slightly different connotation (Ref. 2-5). The preferred terminology for our purposes is to refer to $\bar{\bar{A}}$ as a tensor and to the components as being physical, covariant, contravariant or mixed. This is the system of Ref. 2-5.

2.4.1 Scalar Components of Tensors

There are two ways that the scalar components of tensors can be used. The first way is simply to recognize from the form of tensor involved how many scalar components are necessary and when that number is specified, stop. If a coordinate change is subsequently desirable, these scalar components can be reevaluated in the new coordinate system. There should be a red flag flying at this point for, if scalars are invariant, why do they have different values in different coordinate systems? The reason is that they are not proper scalar invariants but they are scalar components. This point is now illustrated for first- and second-order tensors. This distinction between scalar invariants and components can be made formally, and the resulting scalar invariants can be used to define the tensor. Usually, this is more trouble than it is worth because the scalar components are perfectly adequate to model transport phenomena problems. However, the aforementioned illustration is presented to clarify this point.

If, for example, the physical components of a vector (first-order tensor) are known in a Cartesian and in a cylindrical coordinate system to be

$$\begin{aligned} A &= (AX1)(\bar{U}1) + (AX2)(\bar{U}2) + (AX3)(\bar{U}3) = \\ &(AY1)(\bar{V}1) + (AY2)(\bar{V}2) + (AY3)(\bar{V}3) \end{aligned} \tag{2.40}$$

where for this illustration,

$\bar{V}1$ is the radial direction

$\bar{V}2$ is the angular direction and,

$\bar{V}3$ is the axial direction.

Now AX1, AX2, AX3, AY1, AY2, AY3 are scalar components of the vector but they certainly are functions of the coordinate systems used. What then are the scalar invariants?

2.4.2 The Scalar Invariants of a First Order Tensor

In any coordinate system, the distance between the origin and any point is called the position vector, $\overline{R}$. If a vector $\overline{A}$ is defined at each point in space, it is said to be a function of $\overline{R}$.

$$\overline{A} = \overline{A}\{\overline{R}\} \tag{2.41}$$

Since $\overline{A}$ and $\overline{R}$ are vectors, they are invariant with respect to coordinate systems.

Note that $\overline{A}$ and $\overline{R}$ lie in a plane, the direction of which is defined by the cross product of $\overline{A}$ and $\overline{R}$. The angle between $\overline{A}$ and $\overline{R}$ lies in this plane. To simplify these relationships, consider a "natural coordinate system" consisting of a cylindrical coordinate system where the position vector (also called the radius vector) is taken as the radial coordinate and also the angular coordinate origin line. The axial coordinate is taken to be parallel to the normal to the plane containing $\overline{A}$ and $\overline{R}$.

Two scalar invariants are immediately obvious: (1) the magnitude of $\overline{A}$, $|\overline{A}|$, and (2) the angle between $\overline{A}$ and $\overline{R}$, θ.

$$\theta = \cos^{-1}\left[\overline{A} \cdot \overline{R}/|\overline{A}||\overline{R}|\right] \tag{2.42}$$

The third scalar invariant must be a number which fixes the direction of the plane containing $\overline{A}$ and $\overline{R}$. There is no obvious choice for such a scalar invariant. It is tempting to use the magnitude of the cross product of $\overline{A}$ and $\overline{R}$ as the third invariant, but it can be calculated from the other two; therefore, it does not supply an independent relationship.

Inability to calculate the third invariant means that the axial direction of the "natural coordinate system" cannot be determined from $\overline{A}$ and $\overline{R}$ alone. Perhaps the direction of the cross product of $\overline{A}$ and $\overline{R}$ may be thought of as the third invariant, but this still does not yield the necessary third equation, unless this direction is assigned an arbitrary value — like zero. Therefore three scalar invariants are not available for a simple algebraic transformation of the components $\bar{A}$ from one coordinate system to another. The formal linear transformation for components of $\bar{A}$ can be accomplished, but there is a more simple way.

If components of $\bar{A}$ and $\bar{R}$ are known in one coordinate system (say the $\bar{U}$, X system) and one wishes to determine the components of $\bar{A}$ in, for example, the $\bar{W}$, Z system, the following relationships can be used.

$$
\begin{aligned}
(\bar{A} \bullet \bar{W}1) &= AZ1 \\
(\bar{A} \bullet \bar{W}2) &= AZ2 \\
(\bar{A} \bullet \bar{W}3) &= AZ3
\end{aligned}
\qquad (2.43)
$$

Of course one must be able to determine $\bar{W}$, Z from $\bar{U}$, X, but transformation laws have already been presented for this determination.

2.4.3 The Scalar Invariants of a Second-Order Tensor

Since the components of a tensor can be defined only after the unit or base vectors of a coordinate system have been chosen, a coordinate system with three unit vectors: $\bar{W}1$, $\bar{W}2$, $\bar{W}3$, is chosen here.

Let $\bar{\bar{B}}$ represent the second order tensor that is being discussed. It is defined in the $\bar{W}$ coordinate system by its "ij" components.

$$
\bar{\bar{B}} = \begin{pmatrix} B11 & B12 & B13 \\ B21 & B22 & B23 \\ B31 & B32 & B33 \end{pmatrix} \qquad (2.44)
$$

The easiest way to represent the scalar invariants of this tensor is to write it as three vectors and then use the vector invariants. To do this, assume that the dot product of $\bar{\bar{B}}$ and an arbitrary unit vector $\bar{H}$ forms a vector which equals a scalar, E, times $\bar{H}$.

$$\bar{\bar{B}} \cdot \bar{H} = E\,\bar{H} \tag{2.45}$$

Let $\bar{\bar{I}}$ be the unit tensor:

$$\bar{\bar{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.46}$$

in the $\bar{W}$ coordinate system.

Therefore,

$$(\bar{\bar{B}} - E\,\bar{\bar{I}}) \cdot \bar{H} = 0 \tag{2.47}$$

This is a classic eigenvalue problem. The solution of this problem would give three eigenvalues: E1, E2, E3 and three eigenvectors:

$$\begin{aligned} \bar{H}1 &= (C11)(\bar{W}1) + (C12)(\bar{W}2) + (C13)(\bar{W}3) \\ \bar{H}2 &= (C21)(\bar{W}1) + (C22)(\bar{W}2) + (C23)(\bar{W}3) \\ \bar{H}3 &= (C31)(\bar{W}1) + (C32)(\bar{W}2) + (C33)(\bar{W}3) \end{aligned} \tag{2.48}$$

where C1i, C2i, C3i are the direction cosines of $\bar{H}$ with respect to $\bar{W}$.

Thus

$$\begin{aligned} \bar{E}1 &= E1\ \bar{H}1 \\ \bar{E}2 &= E2\ \bar{H}2 \\ \bar{E}3 &= E3\ \bar{H}3 \end{aligned} \tag{2.49}$$

are the three vector components of $\bar{\bar{B}}$; however, two of the eigenvalues may be complex, making two of the eigenvectors also complex. To avoid this problem the use of symmetric and antisymmetric tensors is introduced. Note that nine and only nine scalar invariants of $\bar{\bar{B}}$ can be calculated – be they real or complex. Many other scalar quantities can be calculated from these nine scalar invariants; in fact any of these sets of quantities can be designated the scalar invariants of the tensor $\bar{\bar{B}}$.

Any second-order tensor can be represented as the sum of a symmetric tensor ($\bar{\bar{B}}S$) and an antisymmetric tensor ($\bar{\bar{B}}A$).

$$\bar{\bar{B}} = \bar{\bar{B}}S + \bar{\bar{B}}A \tag{2.50}$$

where the components of $\bar{\bar{B}}S$ and $\bar{\bar{B}}A$ in a $\bar{W}$ coordinate system are

$$\begin{aligned} BSij &= (1/2)(Bij + Bji) \\ BAij &= (1/2)(Bij - Bji) \end{aligned} \tag{2.51}$$

Repeating the contraction scheme used on the general tensor $\bar{\bar{B}}$ to reduce it to vector components, let

$$\bar{\bar{B}}S \cdot \bar{H}S = ES\ \bar{H}S$$

or

$$(\bar{\bar{B}}S - ES\ \bar{\bar{I}}) \cdot \bar{H}S = 0 \tag{2.52}$$

The solution of this equation yeilds three eigenvalues (ES1, ES2, ES3) and three eigenvectors

$$\begin{aligned} \bar{H}S1 &= (CS11)(\bar{W}1) + (CS12)(\bar{W}2) + (CS13)(\bar{W}3) \\ \bar{H}S2 &= (CS21)(\bar{W}1) + (CS22)(\bar{W}2) + (CS23)(\bar{W}3) \\ \bar{H}S3 &= (CS31)(\bar{W}1) + (CS32)(\bar{W}2) + (CS33)(\bar{W}3) \end{aligned} \tag{2.53}$$

Hence

$$\overline{ES1} = ES1\ \overline{HS1}$$

$$\overline{ES2} = ES2\ \overline{HS2} \tag{2.54}$$

$$\overline{ES3} = ES3\ \overline{HS3}$$

(Note: If a vector contains a number in its name, there must be a defined unit vector which corresponds to the vector; i.e., the index on ESi corresponds to the ith unit vector ($\overline{HSi}$) not to the ith component of the coordinate vector, $\overline{W}$.) Since the tensor $\overline{\overline{BS}}$ is symmetric all of the eigenvalues ESi's and eigenvectors $\overline{HSi}$'s will be real (Ref. 2.3, p. 61). There are three scalar invariants for each of the $\overline{ESi}$ vectors.

Consider now the antisymmetric tensor.

$$\begin{aligned}\overline{\overline{BA}} = {} & BA11\ \overline{W1}\overline{W1} + BA12\ \overline{W1}\overline{W2} + BA13\ \overline{W1}\overline{W3} + \\ & BA21\ \overline{W2}\overline{W1} + BA22\ \overline{W2}\overline{W2} + BA23\ \overline{W2}\overline{W3} + \\ & BA31\ \overline{W3}\overline{W1} + BA32\ \overline{W3}\overline{W2} + BA33\ \overline{W3}\overline{W3}\end{aligned} \tag{2.55}$$

in general, but $\overline{\overline{BA}}$ is antisymmetric.

Therefore, by definition

$$BA11 = BA22 = BA33 = 0$$

$$BA12 = -BA21$$

$$BA13 = -BA31$$

$$BA23 = -BA32 \tag{2.56}$$

Since $\bar{\bar{B}}A$ has only three components, it may be interpreted as a vector (Ref. 2-6, p. 24).

Let

$$\left.\begin{aligned} BA12 &= F3 \\ BA23 &= F1 \\ BA31 &= F2 \end{aligned}\right\}$$

or

$$Fk = \sum_{i,j=1}^{3} \epsilon ijk \; BAij \tag{2.57}$$

where

$$\begin{aligned} \epsilon ijk &\equiv +1, \text{ if } i,j,k = 1,2,3 \\ &\qquad\qquad\qquad\quad 2,3,1 \\ &\qquad\qquad\qquad\quad 3,1,2 \\ &\equiv -1, \text{ if } i,j,k = 3,2,1 \\ &\qquad\qquad\qquad\quad 2,1,3 \\ &\qquad\qquad\qquad\quad 1,3,2 \\ &\equiv \;\; 0, \text{ if any } i, j, k \text{ are equal} \end{aligned}$$

where

$$\bar{F} = F1\,\bar{W}1 + F2\,\bar{W}2 + F3\,\bar{W}3 \tag{2.58}$$

is called the vector of the antisymmetric tensor. Again $\bar{F}$ has three scalar invariants which were previously discussed.

Two important operations result:

$$\bar{\bar{B}}A \cdot \bar{A} = \bar{A} \times \bar{F} \text{ and } \bar{A} \cdot \bar{\bar{B}}A = \bar{F} \times \bar{A} \tag{2.59}$$

where $\bar{A}$ is an arbitrary vector.

Therefore, the eigenvalue problem for the antisymmetric tensor becomes:

$$\overline{\overline{B}}A \cdot \overline{A} = E\overline{H} = \overline{A} \times \overline{F} \tag{2.60}$$

Therefore

$$E = O,\ i\ |\overline{F}|,\ -i\ |\overline{F}|, \quad \text{where}$$

i is the $\sqrt{-1}$, Ref. (2.3), p. 60.

The most common usage of the split on $\overline{\overline{B}}$ is when the derivative of the velocity with respect to the position vector is divided into the symmetric rate-of-strain tensor and the antisymmetric rotation tensor.

2.4.4 Transformation of Tensor Components

If the components of a tensor are known in one coordinate system, they can be transformed into components in a second coordinate system by the following relationships.

Let one general physical curvilinear coordinate system be designated by $\overline{W}A$, ZA, and $\overline{\overline{G}}A$ and a second by $\overline{W}B$, ZB, and $\overline{\overline{G}}B$. Then for a <u>first-order tensor</u>:

$$\overline{F}\ F = FA1 * \overline{W}A1 + FA2 * \overline{W}A2 + FA3 * \overline{W}A3$$

or

$$\overline{F}\ F = FB1 * \overline{W}B1 + FB2 * \overline{W}B2 + FB3 * \overline{W}B3 \tag{2.61}$$

Where

$$FB1 = \left(\frac{GB11}{GA11}\right)^{1/2} \frac{\partial ZB1}{\partial ZA1}\ FA1 + \left(\frac{GB11}{GA22}\right)^{1/2} \left(\frac{\partial ZB1}{\partial ZA2}\right) FA2$$

$$+ \left(\frac{GB11}{GA33}\right)^{1/2} \left(\frac{\partial ZB1}{\partial ZA2}\right) FA3 \tag{2.62}$$

and FB2, FB3 are given by like relationships, or in general (Ref. 2.2):

$$\mathrm{FBi} = \sum \left(\frac{\mathrm{GBii}}{\mathrm{GAjj}}\right)^{\frac{1}{2}} \left(\frac{\partial \mathrm{ZBi}}{\partial \mathrm{ZAj}}\right) \mathrm{FAj} \tag{2.63}$$

where the summation is on j and an equation is obtained for each i.

Second-order tensor:

$$\bar{\bar{\mathrm{E}}}\mathrm{E} = \sum\sum \mathrm{EAmn} * \overline{\mathrm{W}}\mathrm{Am} * \overline{\mathrm{W}}\mathrm{An} \tag{2.64}$$

where the first sum is on m and the second is on n over the range 1, 2, 3.

$$\mathrm{EBij} = \sum\sum \left[\frac{\mathrm{GBmm}}{\mathrm{GAii}} \; \frac{\mathrm{GBnn}}{\mathrm{GAjj}}\right]^{1/2} \left(\frac{\partial \mathrm{ZBm}}{\partial \mathrm{ZAi}} \; \frac{\partial \mathrm{ZBn}}{\partial \mathrm{ZAj}}\right) \mathrm{EAmn} \tag{2.65}$$

where the first sum is on m and the second is on n over the range 1, 2, 3.

N^th-order tensor:

$$\bar{\bar{\mathrm{D}}} = \sum\sum \cdots \sum \mathrm{DAmn} \ldots \mathrm{s} * \overline{\mathrm{W}}\mathrm{Am} * \overline{\mathrm{W}}\mathrm{An} * \ldots * \overline{\mathrm{W}}\mathrm{As} \tag{2.66}$$

$$\mathrm{DBij}\ldots\ell = \sum\sum\cdots\sum \left[\frac{\mathrm{GBmm}}{\mathrm{GAii}} \; \frac{\mathrm{GBnn}}{\mathrm{GAjj}} \cdots \frac{\mathrm{GBss}}{\mathrm{GA}\ell\ell}\right]^{1/2} \left(\frac{\partial \mathrm{ZBm}}{\partial \mathrm{ZAi}} \; \frac{\partial \mathrm{ZBn}}{\partial \mathrm{ZAj}} \cdots \frac{\partial \mathrm{ZBs}}{\partial \mathrm{ZA}\ell}\right) \mathrm{DAmn}\ldots\mathrm{s} \tag{2.67}$$

If only $\overline{\mathrm{W}}$N, ZN or only $\overline{\mathrm{W}}$Ø, ZØ coordinates are used, the ratios of G's do not appear in these transformation equations because they have been absorbed in the definitions of the base vectors. If Cartesian coordinates are used, the G's do not appear because they are unity.

These transformation laws can also be used to go to or from $\overline{W}N$, $ZN \rightarrow \overline{W}\phi$, $Z\phi$. The G's will appear in that case.

The operator, ∇

$$\nabla = \partial(\quad)/\partial\overline{R}.$$

Before the vector and tensor quantities that have just been defined can be used effectively, their derivatives must also be evaluated.

This may be done thusly:

$$\frac{\partial\overline{A}}{\partial Zj} = \sum \left[Bij + \frac{\partial Ai}{\partial Zj}\right] \overline{W}i \qquad \text{sum on i, giving j equations.} \tag{2.68}$$

The significance of this equation is that the scalar coefficients of the vector represented by the differentiation are defined. Bij is a matrix which is a function of $\overline{\overline{G}}$.

In a similar manner

$$\frac{\partial\overline{\overline{T}}}{\partial Zn} = \sum \frac{\partial Tij}{\partial Zn} \overline{W}i\,\overline{W}j + Tij\,\frac{\partial[\overline{W}i\,\overline{W}j]}{\partial Zn} \tag{2.69}$$

sum on i and j, giving N equations.

$\partial[\overline{W}i\,\overline{W}j]/\partial Zn$ is a function of $\overline{\overline{G}}$.

2.5 CLOSURE

With the terms defined in these pages, two conclusions are possible.

1. If the conservation laws can be written in one coordinate system — even the Cartesian system — they can be written in any other system. All of the unit (or base) vectors can be removed from the equations by writing them as a set of partial differential equations.

2. Before Item 1 can be realized, the proper tensor form of every variable of interest must be known. This is a difficult requirement if empirical data are used, especially for the second order tensors which represent shear-stress and Reynolds stress. The Reynolds stress term will be considered in detail — and the turbulent kinetic energy models will result.

2.6 COMMENTS

(This material is not necessary for continuity of this chapter, but is necessary for completeness.)

A. Implications of $\bar{\bar{G}}$ being symmetric

$$\underline{G12} = G21,\ G13 = \underline{G31},\ \underline{G23} = G32$$

(1)
$$\begin{aligned} \|Gnm\| &= G11\ G22\ G33 + G12\ G23\ G31 + \\ &\quad G31\ G12\ G23 - G31\ G22\ G31 - \\ &\quad -G11\ G23\ G23 - G12\ G12\ G33 \\ &= G11\ G22\ G33 + 2\ G12\ G23\ G31 \\ &\quad -(G31)^2\ G22 - G11\ (G23)^2 - G33(G12)^2 \end{aligned}$$

(2)
$$\begin{aligned} GCF11 &= (G22\ G33 - [G23]^2) \\ GCF21 &= -(G12\ G33 - G23\ G31) \\ GCF31 &= (G12\ G23 - G22\ G31) \\ GFC12 &= -(G12\ G33 - G31\ G23) = GCF21 \\ GCF22 &= (G11\ G33 - [G31]^2) \\ GCF32 &= -(G11\ G23 - G12\ G31) \end{aligned}$$

$$GCF13 = (G12\ \ G23\ -\ G31\ \ G22) = GCF31$$

$$GCF23 = -(G11\ \ G23\ -\ G31\ G12) = GCF32$$

$$GCF33 = (G11\ \ G22\ -\ [G12]^2)$$

(3) Aside-determine what $\overline{G}I$ elements are –

$$\frac{GCF11}{\|Gnm\|} = \frac{G22\ G33\ -[G23]^2}{\left(G11\ (G22\ G33\ -[G23]^2) + 2\ \ G12\ G23\ G31 - \left[(G31)^2\ \ G22\ +\ (G33)(G12)^2\right]\right)} \neq \frac{1}{G11}$$

This inequality would become an equality for orthogonal coordinates.

B. The linear differential transformations were used to relate two different coordinate systems in the previous discussion. But could different transformations be used for the WØ, ZØ system?

For example, usually

$$Z1 = Z1 \{X1,\ X2,\ X3\}$$

$$Z2 = Z2 \{X1,\ X2,\ X3\}$$

$$Z3 = Z3 \{X1,\ X2,\ X3\} \quad \text{are known,}$$

Then

$$dZi = \sum \frac{\partial Zi}{\partial Xj}\ dXj \qquad \text{sum on } j, \text{ giving } i \text{ equations.}$$

Now if this procedure were applied to the calculation of explicit values of the U's, terms involving $\frac{\partial \overline{R}}{\partial XN}$ would have to be evaluated; two problems arise.

(1) An integral expression for $\overline{R}$ in terms of $\overline{W}$, Z would have to be determined; for example, in cylindrical coordinates, $\overline{R} = \overline{V}R * YR + \overline{V}Z * YZ$. The lack of an integral expression for $\overline{R}$ in terms of stretching functions and W's and Z's is a discouraging fact. (2) Simple expressions like

$$\overline{U}i = \sum \frac{\partial Zj}{\partial Xi} \overline{W}j \quad \text{sum on j, giving i equations}$$

would not result, because the $\overline{W}j$'s are functions of Xj's; they are not constants.

C. Evaluation of $\partial(\ )/\partial\overline{R}$ for vectors and second-order tensors.

(1) Definitions and nomenclature from Ref. (2-1).

$$\frac{\partial \overline{e}_i}{\partial X^j} = \left\{ \begin{matrix} m \\ ij \end{matrix} \right\} \overline{e}_m$$

$$\left\{ \begin{matrix} m \\ ij \end{matrix} \right\} = [ij;k]\, g^{mk}$$

$$\left\{ \begin{matrix} m \\ ij \end{matrix} \right\} = \sum_{k=1}^{3} \left(\frac{1}{2} \left[\frac{\partial g_{ik}}{\partial X^j} + \frac{\partial g_{jk}}{\partial X^i} - \frac{\partial g_{ij}}{\partial X^k} \right] g^{mk} \right)$$

$$= \frac{1}{2} \left(g^{m1} \left[\frac{\partial g_{i1}}{\partial X^j} + \frac{\partial g_{j1}}{\partial X^i} - \frac{\partial g_{ij}}{\partial X^1} \right] + g^{m2} \left[\frac{\partial g_{i2}}{\partial X^j} + \frac{\partial g_{j2}}{\partial X^i} - \frac{\partial g_{ij}}{\partial X^2} \right] + g^{m3} \left[\frac{\partial g_{i3}}{\partial X^j} + \frac{\partial g_{j3}}{\partial X^i} - \frac{\partial g_{ij}}{\partial X^3} \right] \right)$$

$$\therefore \frac{\partial \overline{e}_i}{\partial X^j} = \sum_{m=1}^{3} \left\{ \begin{matrix} m \\ ij \end{matrix} \right\} \overline{e}_m .$$

$$= \left(\frac{1}{2}\right)\left(g^{11}\left[\frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^1}\right] + g^{12}\right.$$

$$\left[\frac{\partial g_{i2}}{\partial x^j} + \frac{\partial g_{j2}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^2}\right] + g^{13}$$

$$\left.\left[\frac{\partial g_{i3}}{\partial x^j} + \frac{\partial g_{j3}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^3}\right]\right)\bar{e}_1 +$$

$$f\left(i, j\right) \quad \bar{e}_2 + f\left(i, j\right) \quad \bar{e}_3$$

$i, j = 1, 2; 2, 3; 3, 1;$

$1, 3; 2, 1; 3, 2;$

$1, 1; 2, 2; 3, 3;$

(2) Introduction of general unit vectors

$$\frac{\partial (\bar{W}i\, g_{ii}{}^{1/2})}{\partial x^j} = f(g_{ijm})\ \bar{e}_m = f(g)\,(\bar{W}m\, g_{mm}{}^{1/2})$$

$$\bar{W}i\ \frac{\partial g_{ii}{}^{1/2}}{\partial x^j} + g_{ii}{}^{1/2}\ \frac{\partial \bar{W}i}{\partial x^j} = f(g)\ g_{mm}{}^{1/2}\ \bar{W}m$$

$$\frac{\partial \bar{W}i}{\partial x^j} = -\frac{\partial \ln (g_{ii})^{1/2}}{\partial x^j}\ \bar{W}i + f(g)\ g_{mm}{}^{1/2}\ \bar{W}m$$

where $f(g)$ represents a function of g or g_{ij}.

$$\therefore \quad \frac{\partial \overline{A}}{\partial X^j} = \sum_{i=1}^{3} \quad A_i \; \frac{\partial \overline{W}i}{\partial X^j} + \overline{W}i \; \frac{\partial Ai}{\partial X^j}$$

$$= A1 \frac{\partial \overline{W}1}{\partial X^j} + A2 \frac{\partial \overline{W}2}{\partial X^j} + A3 \frac{\partial \overline{W}3}{\partial X^j} +$$

$$+ \overline{W}1 \frac{\partial A1}{\partial X^j} + \overline{W}2 \frac{\partial A2}{\partial X^j} + \overline{W}3 \frac{\partial A3}{\partial X^j}$$

But

$$\left(\frac{\partial \overline{W}1}{\partial X^j}\right) = \left(B1j\right) \overline{W}1 + \left(B2j\right) \overline{W}2 + \left(B3j\right) \overline{W}3$$

(see Item (3)). B is a matrix function of g_{ij}.

$$\frac{\partial \overline{A}}{\partial X^j} = \left(Bij + \frac{\partial A1}{\partial X^j}\right) \overline{W}1 + \left(B2j + \frac{\partial A2}{\partial X^j}\right) \overline{W}2 +$$

$$\left(B3j + \frac{\partial A3}{\partial X^j}\right) \overline{W}3$$

(3) Derivatives of unit vectors

$$\frac{\partial \overline{W}i}{\partial X^j} = -\left(\frac{\partial\left[\ln \left(g_{ii}\right)^{1/2}\right]}{\partial X^j}\right) \overline{W}i + \frac{1}{2} \left\{ g^{11} \left(g_{11}\right)^{1/2} \left(\frac{\partial g_{i1}}{\partial X^j} + \right.\right.$$

$$+ \frac{\partial g_{ji}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^1}\Bigg) + g^{12}\left(g_{11}\right)^{1/2}\left(\frac{\partial g_{i2}}{\partial x^j} + \frac{\partial g_{j2}}{\partial x^i} -\right.$$

$$\left. - \frac{\partial g_{ij}}{\partial x^2}\right) + g^{13}\left(g_{11}\right)^{1/2}\left(\frac{\partial g_{i3}}{\partial x^j} + \frac{\partial g_{j3}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^3}\right)\Bigg\} \overline{W} +$$

$$+ \frac{g_{22}{}^{1/2}}{2}\Bigg\{ g^{21}\left[\frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^1}\right] + g^{22}$$

$$\left[\frac{\partial g_{i2}}{\partial x^j} + \frac{\partial g_{j2}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^2}\right] + g^{23}\left[\frac{\partial g_{i3}}{\partial x^j} + \frac{\partial g_{j3}}{\partial x^i} -\right.$$

$$\left. - \frac{\partial g_{ij}}{\partial x^3}\right]\Bigg\} \overline{W}2 + \frac{g_{33}{}^{1/2}}{2}\Bigg\{ g^{31}\left[\frac{\partial g_{i1}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^i} -\right.$$

$$\left. - \frac{\partial g_{ij}}{\partial x^1}\right] + g^{32}\left[\frac{\partial g_{i2}}{\partial x^j} + \frac{\partial g_{j2}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^2}\right] +$$

$$g^{33}\left[\frac{\partial g_{i3}}{\partial x^j} + \frac{\partial g_{j3}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^3}\right]\Bigg\} \overline{W}3$$

$$\left[\frac{\partial \overline{W}i}{\partial x^j}\right] = \begin{bmatrix} \frac{\partial \overline{W}1}{\partial x^1} & \frac{\partial \overline{W}1}{\partial x^2} & \frac{\partial \overline{W}1}{\partial x^3} \\ \frac{\partial \overline{W}2}{\partial x^1} & \frac{\partial \overline{W}2}{\partial x^2} & \frac{\partial \overline{W}2}{\partial x^3} \\ \frac{\partial \overline{W}3}{\partial x^1} & \frac{\partial \overline{W}3}{\partial x^2} & \frac{\partial \overline{W}3}{\partial x^3} \end{bmatrix}$$

(4) Extension to second-order tensors.

$$\frac{\partial(\overline{\overline{T}})}{\partial x^n} = \frac{\partial(Tij\ Wi\ Wj)}{\partial x^n} =$$

$$= \left(\frac{\partial Tij}{\partial x^n}\right)\left(\overline{W}i\ \overline{W}j\right) + Tij\ \frac{\partial \overline{W}i\ \overline{W}j}{\partial x^n}$$

$$= \left(\frac{\partial Tij}{\partial x^n}\right)\left(\overline{W}i\ \overline{W}j\right) + Tij\left[\overline{W}i\ \frac{\partial \overline{W}j}{\partial x^n} + \frac{\partial \overline{W}i}{\partial x^n}\ \overline{W}j\right]$$

where n is a particular coordinate and i, j are both summed.

Section 3
TURBULENT MIXING VIA TURBULENT KINETIC ENERGY

If the conservation laws for laminar flows are assumed to be true at each instant in a turbulent flow, formal averaging of these equations can be performed. These operations produce the Reynolds stress-tensor. This section attempts to represent the best possible analysis of these averaged equations before resorting to empirical data fits. Such efforts make extensive use of the turbulent kinetic energy, and have come to be called turbulent kinetic energy models to distinguish them from eddy viscosity models. These methods are still being researched; hence, a definitive technology does not yet exist. Simple cases will be developed and general cases outlined.

The starting place for analysis is the conservation laws. Immediately a nomenclature problem arises. No universally accepted system exists. Facility to read indicial systems must be established. Therefore a system which is flexible enough to serve as a standard and which preserves the summation feature of the indicial system will be established and used throughout. An extensive listing of symbols is given in the Nomenclature, page iv, to facilitate comparisons to standard references.

3.1 THE CONSERVATION LAWS

Consider the following statement of the continuity and momentum equations for a single component fluid with no body forces acting on it. A Cartesian coordinate system $\overline{U}$, X will be used; these coordinates are not moving, i.e., they are inertially fixed.

Continuity equation:

$$\frac{\partial \rho}{\partial \theta} + \sum_{i=1}^{3} \frac{\partial \rho \cdot Vi}{\partial Xi} = 0 \qquad (3.1)$$

ρ is density, θ is time, and $\overline{V}$ is velocity. Notice, our system of using capital letters has been compromised, because Greek letters have also been used to name variables. This was done to conserve space in writing the equation. FORTRAN names (RHØ, THET) could have been used and must be in programming, but they are read with the same name.

Momentum equation:

$$\rho \left(\frac{\partial Vi}{\partial \theta} + \sum_{j=1}^{3} Vj \frac{\partial Vi}{\partial Xj} \right) = \sum_{j=1}^{3} \frac{\partial \tau ji}{\partial Xj} - \frac{\partial P}{\partial Xi} \tag{3.2}$$

τji are the components of the shear stress tensor, $\overline{\overline{\tau}}$. One would like to express $\overline{\overline{\tau}}$ as a function of the velocity field. This functionality is denoted by:

$$\overline{\overline{\tau}} = \overline{\overline{\tau}} \left\{ \overline{V} \right\} \tag{3.3}$$

Since $\overline{\overline{\tau}}$ is a second order tensor and $\overline{V}$ is a vector; this functionality is not straightforward. Consider

$$\frac{\partial \overline{V}}{\partial \overline{R}} = \text{a second-order tensor} \tag{3.4}$$

This tensor may be written in terms of a symmetric and an antisymmetric part. Components of these tensors are:

$$\frac{\partial Vi}{\partial Xj} = \frac{1}{2} \left(\frac{\partial Vi}{\partial Xj} + \frac{\partial Vj}{\partial Xi} \right) + \frac{1}{2} \left(\frac{\partial Vi}{\partial Xj} - \frac{\partial Vj}{\partial Xi} \right) \tag{3.5}$$

It has been previously established that an antisymmetric tensor component is related to a vector such that:

$$Wk = \sum_{i,j=1}^{3} \epsilon ijk \left(\frac{\partial Vi}{\partial Xj} - \frac{\partial Vj}{\partial Xi} \right) \tag{3.6}$$

$\overline{W}$ represents the rotation of a rigid element of fluid; there is no shear associated with such a rotation.

Let

$$Dij = \left(\frac{\partial Vi}{\partial Xj} + \frac{\partial Vj}{\partial Xi} \right) \tag{3.7}$$

This deformation or rate-of-strain tensor is related to shear stress,

by

$$\overline{\overline{\tau}} = \overline{\overline{\tau}} \left\{ \overline{\overline{D}} \right\} \tag{3.8}$$

Let

$$\overline{\overline{\tau}} = A \ \overline{\overline{I}} + B \ \overline{\overline{D}} + C \ \overline{\overline{D}} \cdot \overline{\overline{D}} \tag{3.9}$$

for a Newtonian fluid where

$$A = -\left(\frac{2}{3} \mu - \beta \right) \nabla \cdot \overline{V}, \ B = + \mu, \ C = 0 \tag{3.10}$$

and

$$\overline{\overline{\tau}} = \left[-\left(\frac{2\mu}{3} - \mu B \right) \nabla \cdot \overline{V} \right] \overline{\overline{I}} + \mu \overline{\overline{D}} \tag{3.11}$$

Assuming $\beta = 0$,

$$\left. \begin{aligned} \overline{\overline{\tau}} &= \left[-\frac{2\mu}{3} \nabla \cdot \overline{V} \right] \overline{\overline{I}} + \mu \overline{\overline{D}} \quad \text{or} \\ \tau ij &= \left[-\frac{2\mu}{3} \sum_{k=1}^{3} \frac{\partial Vk}{\partial Xk} \delta ij \right] + \mu Dij \end{aligned} \right\} \tag{3.12}$$

where δij is the Kronecker delta and its properties are:

$$\left.\begin{aligned} \delta ij &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}\right\} \qquad (3.13)$$

Notice $\tau ij = \tau ji$; i.e., $\overline{\overline{\tau}}$ is summetric. Pressure may be considered an additional stressing force and a pressure tensor is defined as

$$\sigma ij = \tau ij - P\,\delta ij = -P\,\delta ij + \mu \left(Dij - \frac{2}{3} \sum_{k=1}^{3} \frac{\partial Vk}{\partial Xk}\,\delta ij \right) \qquad (3.14)$$

Finally, after noting that i and j may be interchanged

$$\frac{\partial \tau ji}{\partial Xm} = \frac{\partial}{\partial Xm}(\mu\ Dji) - \frac{2}{3}\,\frac{\partial}{\partial Xm}\left(\mu \sum_{k=1}^{3} \frac{\partial Vk}{\partial Xk}\ \delta ji \right) \qquad (3.15)$$

The momentum equation becomes,

$$\rho \left(\frac{\partial Vi}{\partial \theta} + \sum_{j=1}^{3} Vj \frac{\partial Vi}{\partial Xj} \right) = \sum_{j=1}^{3} \frac{\partial}{\partial Xj} \left(\mu \left[Dji - \frac{2}{3} \sum_{k=1}^{3} \frac{\partial Vk}{\partial Xk}\ \delta ji \right] \right)$$

$$- \frac{\partial P}{\partial Xi} \qquad (3.16)$$

This is the Navier-Stokes equation (without body forces) as given in Hinze (Ref. 3-1, p.16) and Schlichting (Ref. 3-2, p.61).

Consider the case of constant viscosity. (Details Hinze (Ref. 3-1, p. 17).

$$\rho \left(\frac{\partial Vi}{\partial \theta} + \sum_{j=1}^{3} Vj \frac{\partial Vi}{\partial X_j} \right) = \mu \sum_{j=1}^{3} \frac{\partial^2 Vi}{\partial Xj\ \partial Xj} + \frac{\mu}{3} \frac{\partial}{\partial X_i} \sum_{k=1}^{3} \frac{\partial Vk}{\partial Xk}$$

$$- \frac{\partial P}{\partial Xi} \qquad (3.17)$$

To average Eqs. (3.1) and (3.17), define mean and fluctuating components

$$\overline{V} = \overline{V}M + \overline{V}F, \quad P = PM + PF, \quad \rho = \rho M + \rho F. \tag{3.18}$$

Average and combine said equations.

$$\rho M \left(\frac{\partial VMi}{\partial \theta} + \sum_{j=1}^{3} VMj \frac{\partial VMi}{\partial Xj} \right) = - \frac{\partial PM}{\partial Xi} + \mu \left(\sum_{j=1}^{3} \frac{\partial^2 VMi}{\partial Xj \partial Xj} + \right.$$

$$\left. + \frac{1}{3} \frac{\partial}{\partial Xi} \sum_{k=1}^{3} \frac{\partial VMk}{\partial Xk} \right) - \left(\sum_{j=1}^{3} \frac{\partial}{\partial Xj} \rho M \langle VFi\ VFj \rangle + \frac{\partial}{\partial \theta} \langle \rho F\ VFi \rangle \right.$$

$$\left. + \sum_{j=1}^{3} \left[\frac{\partial}{\partial Xj} \left(\langle \rho F\ VFj \rangle VMi \right) + \frac{\partial}{\partial Xj} \left(\langle \rho F\ VFi \rangle VMj \right) + \frac{\partial}{\partial Xj} \left(\langle \rho F\ VFi\ VFj \rangle \right) \right] \right) \tag{3.19}$$

where $\langle \ \rangle$ denotes the time-average of the included products. Note, values for $\langle VFi\ VFj \rangle$, $\langle \rho F\ VFi \rangle$, $\langle \rho F\ VFi\ VFj \rangle$ must be obtained to solve the momentum equations. Considering the incompressible continuity equation

$$\sum_{j=1}^{3} \frac{\partial Vj}{\partial Xj} = \sum_{j=1}^{3} \left(\frac{\partial VMj}{\partial Xj} + \frac{\partial VFj}{\partial Xj} \right) = 0 \tag{3.20}$$

Therefore

$$\sum_{j=1}^{3} \frac{\partial VFj}{\partial Xj} = 0 = \sum_{j=1}^{3} \frac{\partial VMj}{\partial Xj} \tag{3.21}$$

The incompressible momentum equation is:

$$\rho \left(\frac{\partial VMi}{\partial \theta} + \sum_{j=1}^{3} VMj \frac{\partial VMi}{\partial Xj} \right) = - \frac{\partial PM}{\partial Xi} + \mu \sum_{j=1}^{3} \frac{\partial^2 VMi}{\partial Xj\, \partial Xj} + $$

$$+ \sum_{j=1}^{3} \frac{\partial}{\partial Xj} \left(- \rho \langle VFi\ VFj \rangle \right) \tag{3.22}$$

A mean body force, FMi, may be added to either Eq. (3.16) or Eq. (3.22) if it is needed. Equation (3.22) with the body force is attributed to Reynolds and the term $\langle VFi\ VFj \rangle$ is called the Reynolds stress.

Reynolds stress is a second-order tensor, with three of its terms appearing in each momentum equation. The next step in the analysis of turbulent flows is to represent this tensor in a tractable form.

3.2 THE REYNOLDS STRESS TENSOR

Before more generality is introduced, the analysis of incompressible, constant viscosity flows will be extended. One would wish to express $\langle VFi\ VFj \rangle$ as $f\{\overline{VM}\}$. Let us denote what the requirements for such a representation would mean. Realize that $\langle VFi\ VFj \rangle$ may be considered components of a second-order, symmetric tensor.

1. The Reynolds stress tensor cannot be proportional simply to the velocity of the fluid, because the equations of motion must be invariant under a Galilean transformation. The same turbulent shear stress must result with a fixed body and moving fluid, or vice versa. Hence, $\langle VFi\ VFj \rangle$ may be proportional to

$$\frac{\partial \overline{VM}}{\partial Xi} \quad \text{or} \quad \frac{\partial^2 \overline{VM}}{\partial Xi\, \partial Xj} \tag{3.23}$$

   See Monin and Yaglom (Ref. 3-3, p 371).

2. For isotropic turbulent flows, the extra stress term must behave as an additional pressure term only. (See Ref. 3-3, p 388).

3. $\langle VFi\ VFj \rangle$ is a symmetric second-order tensor; hence whatever model is chosen to represent it must have the same properties as such a tensor.

4. The additional terms which appear in the time-averaged continuity and momentum equations are single-point correlation functions, and as such must obey certain statistical requirements.

These properties are satisfied by the following mathematical models (Ref. 3-3, p 388).

If the fluid is moving as turbulent slug flow; i.e., there are no velocity gradients, the turbulence will be isotropic and should behave as an addition to the thermodynamic pressure. For the Reynolds stress tensor to be isotropic, its diagonal elements must be constants and its off-diagonal elements zero. To represent this behavior:

$$\rho \langle VFi\ VFj \rangle = \frac{1}{3}\ \rho \sum_{k=1}^{3} \langle VFk\ VFk \rangle\ \delta ij = \frac{2}{3}\ \rho\ K\ \delta ij \tag{3.24}$$

where K is the turbulent kinetic energy per unit mass (abbreviated TKE). If the flow has velocity gradients, $\langle VFi\ VFj \rangle$ cannot be simply set equal to a constant times the velocity because each term in an equation must have the same tensor character. But

$$\left(\frac{\partial VMi}{\partial Xj} + \frac{\partial VMj}{\partial Xi} \right),$$

which equals Dij, is a suitable second-order tensor; therefore, let

$$\langle VFi\ VFj \rangle = F\left\{\overline{V}M\right\} \left(\frac{\partial VMi}{\partial Xj} + \frac{\partial VMj}{\partial Xi} \right) \tag{3.25}$$

where $F\{\overline{VM}\}$ is a function of the vector $\overline{VM}$. Since the isotropic limit previously given can be added to Eq. (3.24) without destroying the functional character of the equation

$$\langle VFi\ VFj\rangle = \frac{2}{3} K\ \delta ij + F\{\overline{VM}\}\left(\frac{\partial VMi}{\partial Xj} + \frac{\partial VMj}{\partial Xi}\right) \tag{3.26}$$

$F\{\overline{VM}\}$ may be a scalar, a second-order tensor, or a fourth-order tensor. It cannot be an odd-order tensor because proper contractions cannot be made to maintain the tensor nature of the equations. Even-order tensors of order higher than four when contracted would become identical with second- and fourth-order terms.

What has been discussed are possible values of $\langle VFi\ VFj\rangle$. Presently an analytical method for obtaining these values will be discussed. Empirical determinations of turbulent mixing have been made which fit some of these functional forms. These may be stated as follows.

F is a scalar.

$$\langle VFi\ VFj\rangle = \frac{K}{3}\ \delta ij - L\ K^{1/2}\ Dij \tag{3.27}$$

where L is <u>the</u> mixing length. Usually the first term on the RHS may be assumed negligible (Ref. 3-4, p. 33), then the expression becomes of the form of that suggested by Boussinesq.

F is a second-order tensor.

$$\rho\langle VFi\ VFj\rangle = \frac{2}{3} K\rho\ \delta ij - \frac{1}{2}\left(\rho\ K\right)^{1/2}\sum_{k=1}^{3}\left(Lik\ Dkj + Ljk\ Dki\right) \tag{3.28}$$

This form was suggested by Monin. Lij are components of a symmetric second-order tensor; hence, they represent six length "scales," since their dimensions are in length. The introduction of Lij is useful to interpret the physical phenomena of mixing, but it does not simplify the problem of determining the Reynolds stress tensor, either the six components of $\overline{\overline{L}}$ or $\langle \overline{V}F\ \overline{V}F \rangle$ must be known.

F is a fourth-order tensor.

$$\rho \langle VFi\ VFj \rangle = \frac{2}{3}\ K\ \rho\ \delta ij\ -\ \rho \sum_{k,m=1}^{3} KKijkm\ Dkm \tag{3.29}$$

where $\overline{\overline{K}}K$ must be of the form:

$$KKijkm = \frac{1}{2}\left(Kik\ \delta jm\ +\ Kjk\ \delta im\right) \tag{3.30}$$

No one appears to have used this as an empirical form of eddy transport.

In all of these forms, the approximation of neglecting the first term on RHS of the Reynolds stress tensor is frequently encountered.

Now we shall return to the problem of obtaining the Reynolds stress tensor by analysis.

3.3 TURBULENT KINETIC ENERGY

To obtain expressions for $\langle VFi\ VFj \rangle$, the conservation laws already described may be manipulated until partial differential equations for the components of interest are obtained; these are called dynamic equations. A more satisfying development which results in the same equations may be made by using the statistical properties of the variables of interest; specifically, equations for the moments are determined. Both of these developments will be reviewed.

3.3.1 The Dynamic Equations (Hinze, Ref. 3-1, p 250)

The incompressible momentum equation written in terms of the mean and fluctuating terms is:

$$\frac{\partial VMi}{\partial \theta} + \frac{\partial VFi}{\partial \theta} + \sum_{k=1}^{3} (VMk + VFk) \frac{\partial (VMi + VFi)}{\partial Xk} = -\frac{1}{\rho} \frac{\partial PM}{\partial Xi} - \frac{1}{\rho} \frac{\partial PF}{\partial Xi} + \nu \sum_{k=1}^{3} \frac{\partial^2 \left(VMi + VFi\right)}{\partial Xk \partial Xk} \tag{3.31}$$

Assume steady mean motion, $\frac{\partial VMi}{\partial \theta} = 0$, and add the fluctuating part of the continuity equation, Eq. (3.21), to the momentum equation.

$$\frac{\partial VFi}{\partial \theta} + \sum_{k=1}^{3} \left(VMk \frac{\partial VMi}{\partial Xk} + VFk \frac{\partial VMi}{\partial Xk} + VMk \frac{\partial VFi}{\partial Xk} + VFk \frac{\partial VFi}{\partial Xk} + VFi \frac{\partial VFk}{\partial Xk} \right) = \text{RHS of Eq. (3.22)} \tag{3.32}$$

Combine the last two terms on the LHS and substract the time-average of this equation, Eq. (3.32), to obtain:

$$\frac{\partial VFi}{\partial \theta} + \sum_{k=1}^{3} \left(VFk \frac{\partial VMi}{\partial Xk} + VMk \frac{\partial VFi}{\partial Xk} + \frac{\partial}{\partial Xk} \left[VFi\ VFk - \langle VFi\ VFk \rangle \right] \right) = -\frac{1}{\rho} \frac{\partial PF}{\partial Xi} + \nu \sum_{k=1}^{3} \frac{\partial^2 VFi}{\partial Xk\ \partial Xk} \tag{3.33}$$

There are three equations corresponding to the three momentum equations. Denote another one by j and write it out.

$$\frac{\partial VFj}{\partial \theta} + \sum_{k=1}^{3} \left(VFk \frac{\partial VMj}{\partial Xk} + VMk \frac{\partial VFj}{\partial Xk} + \frac{\partial}{\partial Xk} \left[VFj\ VFk - \langle VFj\ VJk \rangle \right] \right) = -\frac{1}{\rho} \frac{\partial PF}{\partial Xj} + \nu \sum_{k=1}^{3} \frac{\partial^2 VFk}{\partial Xk\ \partial Xk} \tag{3.34}$$

Multiply Eq. (3.33) by VFj and Eq. (3.34) by VFi, then add, and average. Replace the naturally occurring average terms on the RHS of the resulting equation with their equivalents as listed in Hinze, (Ref. 3-1, p 251) to get:

$$\frac{\partial}{\partial \theta}\langle VFi\ VFj\rangle + \sum_{k=1}^{3}\left[\langle VFj\ VFk\rangle \frac{\partial VMi}{\partial Xk} + \langle VFi\ VFk\rangle \frac{\partial VMj}{\partial Xk}\right.$$
$$\left. + \ VMk \frac{\partial}{\partial Xk}\langle VFi\ VFj\rangle\right] = -\sum_{k=1}^{3}\frac{\partial}{\partial Xk}\langle VFi\ VFj\ VFk\rangle$$
$$- \frac{1}{\rho}\left(\frac{\partial}{\partial Xi}\langle PF\ VFj\rangle + \frac{\partial}{\partial Xj}\langle PF\ VFi\rangle\right) +$$
$$+ \frac{1}{\rho}\left\langle PF\left(\frac{\partial VFj}{\partial Xi} + \frac{\partial VFi}{\partial Xj}\right)\right\rangle +$$
$$+ \nu \sum_{k=1}^{3}\left(\frac{\partial^2\langle VFi\ VFj\rangle}{\partial Xk\ \partial Xk} - 2\left\langle\frac{\partial VFi}{\partial Xk}\ \frac{\partial VFj}{\partial Xk}\right\rangle\right) \qquad (3.35)$$

$\langle VFi\ VFj\rangle$ represents the mean value of two velocity components or the elements of the symmetric Reynolds stress tensor. There are six possible unique combinations of i, j; namely, 1, 1; 2, 2; 3, 3; 1, 2; 2, 3; 3, 1. This means that Eq. (3.35) represents six equations.

A contraction (in the terminology of Cartesian tensor analysis) of these six equations is represented by

$$\frac{\partial}{\partial \theta}\sum_{i=1}^{3}\langle VFi\ VFi\rangle + \sum_{i,\,k=1}^{3}\left[2\ VFi\ VFk\ \frac{\partial VMi}{\partial Xk} +\right.$$
$$\left. VMk\ \frac{\partial}{\partial Xk}\langle VFi\ VFi\rangle\right] =$$
$$-\ 2\sum_{i,\,k=1}^{3}\frac{\partial}{\partial Xk}\left\langle\left(\frac{PF}{\rho} + \frac{VFi\ VFi}{2}\right) VFk\right\rangle +$$

$$+ \nu \sum_{i,k=1}^{3} \left(\frac{\partial^2 \langle VFi\ VFi \rangle}{\partial Xk\ \partial Xk} - 2 \left\langle \frac{\partial VFi}{\partial Xk} \frac{\partial VFi}{\partial Xk} \right\rangle \right) \tag{3.36}$$

This equation is obtained by adding the three Eqs. (3.35) for which i, j = 1, 1; 2, 2; 3, 3. (Note that $\sum_{i=1}^{3} \langle VFi\ VFi \rangle$ is K, which is two times the turbulent kinetic energy.)

To use either the six Eqs. (3.35) or Eq. (3.36), now that they have been derived and all of the terms have been identified, the averaged quantities; i.e., those in broken brackets, $\langle\ \rangle$, may be named as new dependent variables. This means that the number of dependent variables may be made somewhat less than the number of previously named variables. For example, $\rho \langle VFi\ VFj \rangle$ may be called ρK, two variables, and that ρ, VFi and VFj (three variables) are not necessary. K is the turbulent kinetic energy.

Attempts to solve for all of the dependent variables at this point, by making empirical correlations when necessary, is called "using a turbulent kinetic energy approach" to represent turbulence. Various attempts to accomplish this task will be reviewed, but first an alternate derivation of Eqs. (3.35) will be outlined.

3.3.2 The Moment Equations (Monin and Yaglom, (Ref. 3-3, p 374)

At best the operations involved in obtaining Eqs. (3.35) are cumbersome, they certainly are not obvious. A more direct approach is to calculate the mean value of an VFi VFj term directly. This can be done by formulating the equations for moments.

Consider n different fluid dynamic variables (dependent variables). Generally, these may be functions of position and time. Denote these variables by the symbol Ci. In general, different Ci's may be functions of different points in space and time. The "n"th moment of these variables is defined by

$$B = \left\langle C_1 \left\{ \overline{R}1, \theta \right\} * C2 \left\{ \overline{R}2, \theta \right\} * C3 \left\{ \overline{R}3, \theta \right\} \ldots * Cn \left\{ \overline{R}n, \theta \right\} \right\rangle \tag{3.37}$$

the $\overline{R}j$ points may be coincident or different.

Reversing the order of averaging and differentiation of Eq. (3.37) with respect to time.

$$\frac{\partial B}{\partial \theta} = \left\langle \frac{\partial C1}{\partial \theta} * C2 * \ldots Cn \right\rangle + \left\langle C1 * \frac{\partial C2}{\partial \theta} * \ldots * Cn \right\rangle + \ldots + \left\langle C_1 * \ldots * \frac{\partial Cn}{\partial \theta} \right\rangle \tag{3.38}$$

B has been written as a scaler; it may be an "n"th order tensor, if the Ci variables are vectors. However, the defining conservation equations are already written as partial differential equations, which means that one may simply consider the vector components as scalers (if no coordinate change is used; i.e., if no vector or tensor component has to be calculated in another coordinate system).

To proceed, one uses Eq. (3.38) to define a "B," the partials with respect to time that appear on the RHS are obtained directly from conservation laws. This is the logic for why the operations in the previous section are reasonable.

The previous section may be summarized by saying that the time-averaged values of the single-point correlation functions, $\langle$ VFi VFj $\rangle$, were formulated as a set of partial differential equations. The correlations were single-point because all of the position vectors in Eq. (3.37) when applied to this problem were identical. The determination of an average value may be described as determining the correlation of the variables involved. More formal definitions of this term can be made (Ref. 3-3, p 228), but these do not contribute to our further understanding of turbulent mixing.

3.3.3 Closure

Returning to the problem of obtaining a solution to either Eqs. (3.35) or (3.36), one sees that an attempt to calculate directly a correlation function leads to difficulties because more correlations are introduced into the equations. Thus more variables than equations result — this causes what is termed the closure problem. Namely, new equations for the unwanted correlation functions must be specified before a solution can be obtained. Reference 3-3, p 377 summarizes the history of recognizing such difficulties.

The next section reviews the better attempts to supply such new equations.

3.4 SOLUTIONS TO THE TKE EQUATIONS

The dynamic equations Eqs. (3.35) which represent the velocity correlations, $\langle VFi\ VFj \rangle$, are one starting place for using empirical turbulence data, but it is not an obvious starting point. However, Prandtl reportedly suggested an equation for turbulent kinetic energy which was "derived by logical reasoning and dimensional analysis," Eckert and Drake (Ref. 3-5, p 369). This equation is similar to the contracted form of the general kinetic energy equation, Eq. (3.36). In any event, nothing more can be accomplished until some empirical information is introduced. There are four ways that this can be accomplished: (1) postulate enough information to solve Eqs. (3.35); (2) solve Eq. (3.36) or one similar to it for K, using empirical data as necessary; (3) repeat method (2) except solve for both K and an additional length scale, L, from equations like Eq. (3.36); (4) solve Eqs. (3.35) for the components of the Reynolds stress tensor and one equation like Eq. (3.36) for a length scale. Without posing the question why, let us examine what these four methods would entail.

3.4.1 Solutions for all Correlations

In order to understand turbulent kinetic energy methods, consider the work reported in Ref. 3-6, which is a somewhat more general account of that reported in Refs. 3-7 and 3-8.

Rearranging Eqs. (3.35) and using Eq. (3.21)

$$\frac{\partial}{\partial \theta}\langle VFi\ VFj\rangle + \sum_{k=1}^{3} \frac{\partial}{\partial Xk}\left[\left(VMk\langle VFi\ VFj\rangle\right) + \langle VFi\ VFj\ VFk\rangle\right] =$$

$$= -\sum_{k=1}^{3}\left[\langle VFj\ VFk\rangle \frac{\partial VMi}{\partial Xk} + \langle VFi\ VFk\rangle \frac{\partial VMj}{\partial Xk}\right] + \tag{3.39}$$

$$+ \left[\text{The last six terms on the RHS of Eqs. (3.35)}\right]$$

The first pair of these last six terms can be moved to the LHS of our new equation. The second pair are to be left on the RHS. The last pair contain a term that with the definitions on p 251 of Ref. 3-1 can be written:

$$\nu \left\langle \sum_{k=1}^{3} VFj\ \frac{\partial^2 VFi}{\partial Xk\ \partial Xk} \right\rangle \quad \text{which is}$$

$$= \left\langle \sum_{k=1}^{3} VFj\ \frac{\partial}{\partial Xk}(+\ \tau ki) \right\rangle$$

But

$$+\ \frac{\partial (VFj\ \tau ki)}{\partial Xk} = VFj\ \frac{\partial \tau ki}{\partial Xk} + \tau ki\ \frac{\partial VFj}{\partial Xk}$$

Therefore all of these operations can be combined to give:

$$\frac{\partial \langle VFi\ VFj \rangle}{\partial \theta} + \sum_{k=1}^{3} \frac{\partial}{\partial Xk} \Big[VMk \langle VFi\ VFj \rangle + \langle VFi\ VFj\ VFk \rangle +$$

$$+ VFi\ (\delta kj\ PF - \tau kj) + VFj\ (\delta ki\ PF - \tau ki) \Big] =$$

$$= -\sum_{k=1}^{3} \left[\langle VFj\ VFk \rangle \frac{\partial VMi}{\partial Xk} + \langle VFi\ VFk \rangle \frac{\partial VMj}{\partial Xk} \right] +$$

$$+ \frac{1}{\rho} \left\langle PF \frac{\partial VFj}{\partial Xi} \right\rangle + \frac{1}{\rho} \left\langle PF \frac{\partial VFi}{\partial Xj} \right\rangle -$$

$$- \sum_{k=1}^{3} \left(\left\langle \tau ki \frac{\partial VFj}{\partial Xk} \right\rangle + \left\langle \tau kj \frac{\partial VFi}{\partial Xk} \right\rangle \right) \tag{3.40}$$

Since Eqs. (3.40) have introduced new correlation terms, models must be used to relate all of the terms on the RHS to $\langle VFi\ VFj \rangle$ and to mean flow properties. Mellor and Herring (Ref. 3-6) assumed, on the basis of physical arguments, that:

$$\langle VFi\ \tau kj \rangle + \langle VFj\ \tau ki \rangle = \nu \left(\frac{\partial \langle VFj\ VFk \rangle}{\partial Xi} + \frac{\partial \langle VFk\ VFi \rangle}{\partial Xj} + \frac{\partial \langle VFi\ VFj \rangle}{\partial Xk} \right) \tag{3.41}$$

$$\left\langle \tau kj \frac{\partial VFi}{\partial Xk} \right\rangle + \left\langle \tau ki \frac{\partial VFj}{\partial Xk} \right\rangle = \frac{2}{3} \frac{K^{3/2}}{L4} \delta ij \tag{3.42}$$

Where L4 is a function of ν only near a wall.

$$\frac{1}{\rho} \left\langle PF \frac{\partial VFi}{\partial Xj} \right\rangle = -\frac{1}{6} \frac{K}{L1}^{1/2} \left(\langle VFi\ VFj \rangle - \frac{\delta ij}{3} K \right) \tag{3.43}$$

$$\langle PF\ VFi \rangle = \frac{K^{1/2}\ L2}{2}\ \frac{\partial K}{\partial Xi} \tag{3.44}$$

$$\langle VFi\ VFj\ VFk \rangle = K^{1/2}\ L3 \left(\frac{\partial \langle VFj\ VFk \rangle}{\partial Xi} + \right.$$

$$\left. + \frac{\partial \langle VFi\ VFk \rangle}{\partial Xi} + \frac{\partial \langle VFi\ VFj \rangle}{\partial Xk} \right) \tag{3.45}$$

With these five assumed functions (Eqs. (3.41) through (3.45)), Eq. (3.40) becomes:

$$\frac{D\langle VFi\ VFj \rangle}{D\theta} = \sum_{k=1}^{3} \left(\frac{\partial}{\partial Xk} \left[\left(\nu + K^{1/2}\ L3 \right) \left(\frac{\partial \langle VFj\ VFk \rangle}{\partial Xi} + \right. \right. \right.$$

$$\left. \left. + \frac{\partial \langle VFk\ VFi \rangle}{\partial Xj} + \frac{\partial \langle VFi\ VFj \rangle}{\partial Xk} \right) \right] + \frac{K^{1/2}\ L2}{2} \left(\right.$$

$$\left(\delta kj\ \frac{\partial K}{\partial Xi} + \delta ki\ \frac{\partial K}{\partial Xj} \right) - \langle VFk\ VFi \rangle \frac{\partial VMj}{\partial Xk} -$$

$$\left. - \langle VFk\ VFj \rangle \frac{\partial VMi}{\partial Xk} \right) - \frac{1}{3}\ \frac{K^{1/2}}{L1} \left(\langle VFi\ VFj \rangle - \right.$$

$$\left. - \frac{\delta ij}{3}\ K \right) - \frac{2}{3}\ \frac{K^{3/2}}{L4}\ \delta ij \tag{3.46}$$

The L1, L2, L3, L4 are empirical functions which must be specified.

Notice that Eqs. (3.46) can be contracted; i.e., the three equations for i, j = 1, 1; 2, 2; 3, 3 are added, and divided by two to give:

$$\frac{D(K/2)}{D\theta} = \sum_{k=1}^{3} \left(\frac{\partial}{\partial Xk} \left[\left(\nu + K^{1/2}\ L3 \right) \left(\frac{\partial (K/2)}{\partial Xk} + \right. \right. \right.$$

$$+ \sum_{i=1}^{3} \frac{\partial\langle VFi\ VFk\rangle}{\partial Xi}\Bigg) + \quad K^{1/2}\, L2\ \frac{\partial(K/2)}{\partial Xk}\Bigg] -$$

$$- \sum_{i=1}^{3} \langle VFk\ VFi\rangle\ \frac{\partial VMi}{\partial Xk}\Bigg) - \frac{K^{3/2}}{L4} \tag{3.47}$$

There is not an immediate application for this equation, since it represents less information than Eqs. (3.46). Equation (3.47) is known as the turbulent kinetic energy equation.

The six equations represented by Eqs. (3.46) together with the mean continuity equation, Eq. (3.21), and the three mean momentum equations, Eqs. (3.22), can be solved for the ten unknowns: VM1, VM2, VM3, PM, $\langle VF1\ VF1\rangle$, $\langle VF2\ VF2\rangle$, $\langle VF3\ VF3\rangle$, $\langle VF1\ VF2\rangle$, $\langle VF2\ VF3\rangle$, $\langle VF3\ VF1\rangle$. if the necessary boundary conditions are specified. This constitutes a solution to the closure problem.

3.4.2 Boundary-Layer Applications

Generally, the Li's and the necessary boundary conditions are not known; hence, further progress is made by restricting the class of flows one tries to describe. Remember the restrictions to this point are: (1) steady mean flow, (2) constant viscosity, and (3) incompressible.

The largest subclass of flows which have been investigated is two-dimensional thin-shear layers. Notice these flows are not necessarily boundary layers. For this class of flows, let: VM3 = 0 and the primary flow direction be X1. Hence,

$$\frac{\partial VM1}{\partial X1} + \frac{\partial VM2}{\partial X2} = 0 \tag{3.48}$$

$$\frac{DVM1}{D\theta} = -\frac{1}{\rho}\frac{\partial PM}{\partial X1} + \frac{\partial}{\partial X2}\left(-\langle VF1\ VF2\rangle + \nu\frac{\partial VM1}{\partial X2}\right) \tag{3.49}$$

$$\frac{D\langle VF1^2\rangle}{D\theta} = \frac{\partial}{\partial X2}\left[\left(\nu + K^{1/2}\ L3\right)\frac{\partial\langle VF1^2\rangle}{\partial X2}\right] - \frac{1}{3}\frac{K^{1/2}}{L1}\left(VM1^2 - \frac{K}{3}\right) - 2\langle VF1\ VF2\rangle\frac{\partial VM1}{\partial X2} - \frac{2\ K^{3/2}}{3L4} \tag{3.50}$$

$$\frac{D\langle VF2^2\rangle}{D\theta} = \frac{\partial}{\partial X2}\left[3\left(\nu + K^{1/2}\ L3\right)\frac{\partial\langle VF2^2\rangle}{\partial X2} + K^{1/2}L2\frac{\partial K}{\partial X2}\right] - \frac{K^{1/2}}{3\ L1}\left(VM2^2 - \frac{K}{3}\right) - \frac{2\ K^{3/2}}{3\ L4} \tag{3.51}$$

$$\frac{D\langle VF3^2\rangle}{D\theta} = \frac{\partial}{\partial X2}\left[\left(\nu + K^{1/2}\ L3\right)\frac{\partial\langle VF3^2\rangle}{\partial X2}\right] - \frac{K^{1/2}}{3\ L1}\left(VM3^2 - \frac{K}{3}\right) - \frac{2\ K^{3/2}}{3\ L4} \tag{3.52}$$

$$\frac{D\langle VF1\ VF2\rangle}{D\theta} = \frac{\partial}{\partial X2}\left[2\left(\nu + K^{1/2}\ L3\right)\frac{\partial\langle VF1\ VF2\rangle}{\partial X2}\right] - \langle VF2^2\rangle\frac{\partial VM1}{\partial X2} - \frac{K^{1/2}}{3\ L1}\langle VF1\ VF2\rangle \tag{3.53}$$

Donaldson and Rosenbaum (Ref. 3-9) obtained equations quite similar to these. The variations (Ref. 3-9) introduced were: L1 = L3, a different L4 and L2, replacing $K^{3/2}$ with $\langle VFi^2\rangle$ in Eqs. (3.50 and 3.52) and added an $\frac{\langle VF1\ VF2\rangle}{L4}$ term to Eq. (3.53). These minor differences were due to using slightly different assumptions to remove the unwanted correlation terms from the dynamic equations. Equations 3.48 through 3.53 were solved by Ref. 3-9 for specified initial conditions and zero correlation coefficients at the wall and at the edge of the boundary layer. Pressure gradients along the boundary layer were also specified. Note that the specification of the Li functions and the boundary conditions made this a boundary layer analysis instead of a free shear layer analysis; i.e., the basic equations were not changed.

The interesting feature of this boundary layer solution was that properties not easily predicted were calculated (Ref. 3-10). Specifically, transition was calculated for arbitrary perturbations imposed on an initially laminar boundary layer.

3.4.3 Solutions for the Contracted TKE Equation

Recognizing that solving Eqs. (3.48) through (3.53) was not a trivial task, Mellor and Herring (Ref. 3-9) argued that a nearly isotropic turbulence could be defined by a limiting form of Eqs. (3.46) and (3.47), namely:

$$\langle VFi\ VFj \rangle = \delta ij\ \frac{K}{3} - K^{1/2}\ L1 \left(\frac{\partial VMj}{\partial Xi} + \frac{\partial VMi}{\partial Xj} \right) \tag{3.54}$$

and

$$\frac{D(K/2)}{D\theta} = \sum_{k=1}^{3} \left(\frac{\partial}{\partial Xk} \left[\left(\frac{5}{3}\,\nu + \frac{5}{3}\,K^{1/2}\ L3 + K^{1/2}\ L2 \right) \frac{\partial (K/2)}{\partial Xk} \right] - \langle VFk\ VFi \rangle \frac{\partial VMi}{\partial Xk} \right) - \frac{K^{3/2}}{L4} \tag{3.55}$$

This is gratifying, since an equation like Eq. (3.54) has already been postulated as a mathematically possible Reynolds stress model. Equations (3.54) and (3.55) also supply a satisfactory solution to the closure problem, again providing the Li's are specified.

The two-dimensional thin shear layer version of Eqs. (3.54) and (3.55) is

$$\langle VF1\ VF2 \rangle = -\ K^{1/2}\ L1\ \frac{\partial VM1}{\partial X2} \tag{3.56}$$

and

$$\frac{D(K/2)}{D\theta} = \frac{\partial}{\partial X2}\left(\left[K^{1/2}(L2 + \tfrac{5}{3}L3) + \frac{5\nu}{3}\right]\frac{\partial (K/2)}{\partial X2}\right) +$$

$$+\ K^{1/2}\ L1\left(\frac{\partial VM1}{\partial X2}\right)^2 - \frac{K^{3/2}}{L4} \tag{3.57}$$

From numerical experiments in Ref. 3-6

$$L1 \cong L2 + \frac{5}{3}\ L3 \tag{3.58}$$

Assume ν negligible.

$$\frac{D(K/2)}{D\theta} = \frac{\partial}{\partial X2}\left(\left[K^{1/2}\ L1\right]\frac{\partial (K/2)}{\partial X2}\right) +$$

$$+\ K^{1/2}\ L1\left(\frac{\partial VM1}{\partial X2}\right)^2 - \frac{K^{3/2}}{L4} \tag{3.59}$$

Equations (3.56) and (3.59) are precisely the same as those obtained by investigators who postulated physical models for the conservation law controlling the supply of turbulent kinetic energy (Ref. 3-5, p.366).

Now expressions for the Li's must be determined. References 3-6 and 3-9 merely assumed them. Assuming L1 proportional to L4 and recognizing that $(K^{3/2}/L4)$ represents the decay rate of turbulent energy, E, Rotta (Ref. 3-8) suggested using an additional transport equation to calculate this decay rate. Rather than use strictly empirical information to describe L4, Rotta (Ref. 3-11) defined L4 in terms of a two-point correlation function. Upon approximating this correlation function for boundary-layer type flow, a decay rate was defined and a transport equation for such a decay rate was written. A similar equation was used by Spalding and associates (Ref. 3-12).

$$\frac{DE}{D\theta} = C1 \sum_{j,k=1}^{3} \frac{\partial}{\partial Xj} \left[\frac{K}{E} \langle VFj\ VFk \rangle \frac{\partial E}{\partial Xk} \right] - C2 \sum_{j,k=1}^{3} \left(\langle VFj\ VFk \rangle \frac{E}{K} \frac{\partial VMj}{\partial Xk} \right) - C3\ E^2/K \tag{3.60}$$

Where Ci's are empirically determined constants.

Boundary conditions must be specified for Eq. (3.60) as well as all other equations used to represent turbulent transport. Seldom are such conditions known, but both Refs. 3-9 and 3-11 indicate that reasonably obvious and simple boundary condition estimation leads to valid flowfield representations. Of course, better answers can be obtained if exact boundary conditions are known.

In summary, the analysis of turbulent kinetic energy which was made by Rotta (Refs. 3-7 and 3-8) formed the basis of a very powerful tool for describing turbulent transport. Unfortunately, this work was not utilized by English-reading investigators until Bradshaw's (Ref. 3-13) and Glushko's (Ref. 3-14) work became known. A convenient review and slight extension of these works is Ref. 3-15. Since then three types of turbulence models have evolved:

1. Equation (3.55) is solved for K for algebraicly specified length scales, the Li's.

2. Equation (3.55) is again solved for K and hence, Eq. (3.60) is used to calculate E; no Li's are necessary.

3. Equations like (3.46) are solved for the components of $\langle VFi\ VFj \rangle$ with either all Li's specified or Eq. (3.60) being used for L4 and the others specified.

Models 1 and 2 offer no material advantage over eddy viscosity models, although they are not much more computationally difficult and they may be somewhat more accurate.

Model 3 is extremely useful because it offers flexibility to estimate certain effects with only a minimum of specified boundary conditions. However, general computation schemes utilizing this technique have not been extensviely investigated.

Turbulent kinetic energy models may be used to represent both boundary-layers (Ref. 3-16) and free shear layers (Ref. 3-17), provided that the appropriate empirical information for either such flow is utilized. A "best" model has not yet been established, nor is it likely to be. The type of choices of modeling terms is illustrated by the discussion of Harsha (Ref. 18, p. 357).

It should be remembered the variable density flows and flows for which non-Cartesian coordinate systems are desirable have not yet been discussed; they will be in subsequent sections. Note also that for the description of incompressible flows, the introduction of the conservation of energy equation is not necessary.

3.5 FLOWS WITH VARIABLE DENSITY AND COMPOSITION

The procedures which were used to analyze single-component, incompressible flows can conceptually be extended to handle multicomponent, compressible flows. The effect of compressibility in a chemically well-behaved flow probably will be predictable in the near future; several successful researches have already been reported in this area. However, the inclusion of complex chemical effects in turbulent kinetic energy models is a long way from being accomplished, and it is probably not worth the effort to try to do so.

Density can be described with mean ρM and fluctuating, ρF, components for a single-component fluid. An analysis using either the philosophy of calculating the dynamic equations or simply a direct determination of all of the moments of interest would result in a set of ten equations; six like those in Eqs. (3.35) except for some additional terms on the RHS and four new ones for the variables $\langle \rho F \; VFi \rangle$ and $\langle \rho F \; \rho F \rangle$. If the thermal equation of state is used, temperature, both mean, TM, and fluctuating, TF, may be used to replace the density variables. Donaldson et al., (Ref. 3-19) report a set of dynamic equations and then restrict the set to a simple enough atmospheric shear flow that a closed set of turbulent correlation equations is obtained. Initial conditions are postulated and the correlation and conservation equations are solved.

Laster (Ref. 3-20) derives the dynamic momentum and continuity equations, uses a Crocco type energy equation, and restricts the dynamic equations to a boundary layer type flow to obtain closure. The empirical models assumed cause Laster's model to be hyperbolic – a situation to be avoided if possible. Heyman (Ref. 3-21) proposed a somewhat similar model, but one which was much easier to use. Heyman admits extensive use of empirical data, but nevertheless reports some very impressive data comparisons.

Another alternative is that the effects of density fluctuations can be ignored and then models based on incompressible flows can be used directly. References 3-12, 3-22 and 3-23 report good data comparisons using this philosophy; this is not too surprising. Reference 3-1 p 19, suggested such slight dependence based on order of magnitude estimates.

Reference 3-24 is a most interesting paper which formulates a very complete set of dynamic equations. Not only are energy and composition fluctuations considered, but also magnetic and electrical. This formulation assumes a boundary-layer type flow; hence, the final set of closed equations looks like the kinetic energy equation, Eq. (3.36), for which empericial length scales are to be supplied. However, they obtained extra equations for thermal and mass diffusion; these equations contained parameters analogous to the

F

length scales. They concluded that there were insufficient available data to estimate these necessary empirical parameters.

3.6 FLOW DESCRIPTIONS WITH NON-CARTESIAN COORDINATES

The real benefit derived from using turbulent kinetic energy models is that by properly modeling the Reynolds stress tensor, empirical data taken in one flow geometry can be used directly to describe a different flow geometry. As the discussion in Section 2 implies this is not a particularly easy task to accomplish. Furthermore, very few problems are worked through to their solution in general coordinates; i.e., if there are errors in the general coordinate formulation, they might be "washed out" in a simple problem. Still, an attempt at a general coordinate formulation has been reported (Ref. 3-9).

An important subclass of coordinates is a cylindrical coordinate system with axial symmetry. The dynamic equations are appreciably simpler to formulate in such a system, but, if one's attention is limited to two-dimensional, thin shear layer flows, the physical development of the turbulent kinetic energy equation can be easily accomplished (see Refs. 3-12 and 3-18). In fact, the general dynamic equations referenced in the previous paragraph have also been reduced to those for an axial symmetric coordinate system (Ref. 3-25). The turbulent kinetic energy models suggested in Refs. 3-12 and 3-22 are valid for plane and axisymmetric flow.

Some details of the general coordinate formulation are now reviewed to indicate the type of difficulties encountered in such studies. The Ref. 3-9 formulation of the dynamic equations is in indicial rotation. These equations are written with the nomenclature of Section 2. They are, for incompressible flow:

$$\rho \frac{\partial}{\partial \theta} \langle \mathrm{VF\emptyset i\ VF\emptyset k} \rangle + \rho \sum_{j=1}^{3} \mathrm{VMNj} \frac{\partial}{\partial \mathrm{Zj}} \langle \mathrm{VF\emptyset i\ VF\emptyset k} \rangle =$$

$$-\rho \sum_{j=1}^{3} \langle VFNj\ VF\phi k\rangle \frac{\partial VM\phi i}{\partial Zj} - \rho \sum_{j=1}^{3} \langle VFNj\ VF\phi i\rangle$$

$$\frac{\partial VM\phi k}{\partial Zj} - \rho \sum_{j=1}^{3} \frac{\partial}{\partial Zj} \langle VFNj\ VF\phi i\ VF\phi k\rangle$$

$$- \frac{\partial}{\partial Zk} \langle PF\ VF\phi i\rangle - \frac{\partial}{\partial Zi} \langle PF\ VF\phi k\rangle +$$

$$+ \left\langle PF \left(\frac{\partial VF\phi i}{\partial Zk} + \frac{\partial VFOk}{\partial Zi} \right) \right\rangle + \mu \sum_{m,n=1}^{3} GImn$$

$$\frac{\partial^2}{\partial Zm\ \partial Zn} \left\langle VF\phi i\ VF\phi k \right\rangle - 2\mu$$

$$\sum_{m,n=1}^{3} GImn \left\langle \frac{\partial VF\phi i}{\partial Zm} \frac{\partial VF\phi k}{\partial Zn} \right\rangle \qquad (3.61)$$

Notice: (1) Coordinates Z, ZN, Zϕ are taken as being identical; (2) both VFNi and VFϕi components of vectors are used in these equations; (3) due to symmetery, six equations are represented by Eqs. (3.61); and (4) Eqs. (3.61) are in terms of the covariant components of the Reynolds stress tensor. To use Eqs. (3.61), one would first rewrite them in terms of the physical components of the Reynolds stress tensor. However, since Ref. 3-9 also modeled the correlation terms in the RHS of Eq. (3.61), let us simply study their result – in covariant form. Also the continuity and momentum equations are written for completeness.

$$\sum_{j=1}^{3} \frac{\partial VMj}{\partial Zj} = 0 \qquad (3.62)$$

$$\rho \frac{\partial VMi}{\partial \theta} + \rho \sum_{j=1}^{3} VMj \frac{\partial VMi}{\partial Zj} = -\frac{1}{\rho} \frac{\partial PM}{\partial Zi} +$$

$$\sum_{j=1}^{3} \frac{\partial}{\partial Zj}\left(\tau N\emptyset ji \ - \rho \langle VFNj\ VF\emptyset i\rangle\right) \tag{3.63}$$

$$\frac{\partial}{\partial \theta} \langle VF\emptyset i\ VF\emptyset k\rangle + \sum_{j=1}^{3} VMNj \frac{\partial}{\partial Zj} \langle VF\emptyset i\ VF\emptyset k\rangle = -$$

$$- \sum_{j=1}^{3} \langle VFNj\ VF\emptyset k\rangle \frac{\partial VM\emptyset i}{\partial Zj} - \sum_{j=1}^{3} \langle VFNj\ VF\emptyset i\rangle \frac{\partial VM\emptyset k}{\partial Zj}$$

$$+ \sum_{j=1}^{3} \frac{\partial}{\partial Zj}\left[L4\langle VFNm\ VF\emptyset m\rangle^{1/2} \left(GIjm \frac{\partial \langle VF\emptyset i\ VF\emptyset k\rangle}{\partial Zm} + \right.\right.$$

$$\left.\left. \frac{\partial}{\partial Zk} \langle VFNj\ VF\emptyset i\rangle + \frac{\partial}{\partial Zi} \langle VFNj\ VF\emptyset k\rangle \right)\right] +$$

$$\frac{\partial}{\partial Zk}\left[L4 \sum_{m=1}^{3} \langle VFNm\ VF\emptyset m\rangle^{1/2} \sum_{n=1}^{3} \frac{\partial \langle VFNn\ VF\emptyset i\rangle}{\partial Zn}\right] +$$

$$\frac{\partial}{\partial Zi}\left[L4 \sum_{m=1}^{3} \langle VFNm\ VF\emptyset m\rangle^{1/2} \sum_{m=1}^{3} \frac{\partial}{\partial Zn} \langle VFNn\ VF\emptyset k\rangle\right] +$$

$$+ \sum_{m=1}^{3} \frac{\langle VFNm\ VF\emptyset m\rangle^{1/2}}{L4} \left(Gik \sum_{m=1}^{3} \frac{\langle VFNm\ VF\emptyset m\rangle}{3} - \right.$$

$$\left. \langle VF\emptyset i\ VF\emptyset k\rangle \right) + \nu \sum_{m,n=1}^{3} GImn \frac{\partial}{\partial Zm \partial Zn} \langle$$

$$\langle VF\emptyset i\ VF\emptyset k\rangle - \frac{2\nu}{(L5)^2} \langle VF\emptyset i\ VF\emptyset k\rangle \tag{3.64}$$

The significant feature is that all geometric effects have been removed once the length scales are determined in any <u>one</u> experiment. Note that the need for the introduction of physical components still exists in Eqs. (3.64).

3.7 SUMMARY

The axial symmetric model for thin, two-dimensional shear layers is probably the only one that will be used in the near future. Truly complex three-dimensional turbulent flows can be modeled with Eqs. (3.62) through (3.64), if sufficient theoretical or experimental work is ever accumulated to evaluate L4 and L5. A synopsis of this section is given in Table 3-1 by listing a set of typical TKE models.

Table 3-1

TYPICAL TKE MODELS FOR FREE SHEAR LAYERS

1. The Prandtl Energy Model* (Ref. 3-12)

$$-\langle VF1\ VF2\rangle = 0.08\left(\frac{K}{2}\right)^{1/2}\left[0.625\ XG2\right]\frac{\partial VM1}{\partial X2} \qquad (3.65)$$

where XG2 is a lateral distance between the dimensionless 0.1 and 0.9 velocity levels in the free shear layer.

$$\rho\,\frac{D(K/2)}{D\theta} = \frac{1}{X2}\left[\frac{\partial}{\partial X2}\,\frac{\epsilon}{0.7}\,\frac{\partial(K/2)}{\partial X2}\right] + \epsilon\left(\frac{\partial VM1}{\partial X2}\right)^2 - \frac{\rho}{L4}\left(\frac{K}{2}\right)^{3/2}\Big/(0.625XG2) \qquad (3.66)$$

where

$$\epsilon = 0.08\ \rho\ (K/2)^2/E \qquad (3.67)$$

2. Energy Dissipation Model* (Ref. 3-12)

$$-\rho\ \langle VF1\ VF2\rangle = \epsilon\ \frac{\partial VM1}{\partial X2} \qquad (3.68)$$

$$\rho\ \frac{DK}{D\theta} = \frac{\partial}{\partial X2}\left[\frac{\epsilon}{0.7}\ \frac{\partial K}{\partial X2}\right] + 2\ \epsilon\left(\frac{\partial VM1}{\partial X2}\right)^2 - 2\ \rho\ E \qquad (3.69)$$

$$\rho\ \frac{DE}{D\theta} = \frac{\partial}{\partial X2}\left(\frac{\epsilon}{1.3}\ \frac{\partial E}{\partial X2}\right) + 2.86\ \frac{E}{K}\ \epsilon\left(\frac{\partial VM1}{\partial X2}\right)^2 - 3.84\left(\frac{\rho\ E}{K}\right)^2 \qquad (3.70)$$

3. Dynamic Equations with algebraic length scales, Ref. 3-9.

4. Dynamic Equations with another partial differential equation for the decay of turbulent kinetic energy (Ref. 3-12)

*Axisymmetric versions of 1 and 2 are also reported.

Section 4
EDDY VISCOSITY MODELS

4.1 FREE SHEAR LAYERS

The prediction of shear layer properties for reacting flows requires empirical eddy viscosity coefficient models be used in order to remain within practical limits. Even then only parallel flows can be described. In the process of solving problems such as plume afterburning and jet engine or rocket plume dissipation, a working principle has evolved for selecting among the many eddy viscosity models the ones to be applied to particular types of shear layers.

An eddy viscosity model, which would treat all the mixing problems encountered, is desirable. This goal has not been achieved. If a single model must be used for all problems, a choice could of course be made, but in certain regions or for certain types of flows, the knowledge that a different eddy viscosity model performs more realistically will invariably mean that the more realistic model will be used to treat that particular region or problem. For that reason discussion of empirical eddy viscosity models should be based upon the kinds of problems a model will be used to solve.

The eddy viscosity models described will be of the following form

$$\rho\epsilon = C L \left(f \left\{ \rho U \right\} \right) \tag{4.1}$$

where ρ is the density, ϵ is the turbulent kinematic eddy viscosity, C is an empirically determined constant, L is a length characteristic of the mixing and the $f\{\rho U\}$ implies that the eddy viscosity models are functions of the mass flux. Laminar mixing can be represented by a special form of this mixing model; i.e.,

$$\rho\epsilon = C_1 \tag{4.2}$$

where C_1 at most can be a function of temperature and composition.

The choice of which form, turbulent or laminar, is determined by the particular state of the flow being calculated.

4.1.1 Planar Mixing

An eddy viscosity model of the Prandtl type is used to describe the shear layer which develops between two planar streams, 1 and 2.

$$\rho\epsilon = 0.0011\,\Delta t_m \left|\rho_1 U_1 - \rho_2 U_2\right| \tag{4.3}$$

where Δt_m is the thickness of the mixing region. The outer edge of the mixing region is defined as the position where the mass flux ratio of successive stream function positions, $(\rho_i U_i)/\rho_{i+1} U_{i+1})$ is 0.999. The inner edge is defined similarly using the ratio $(\rho_{i+1} U_{i+1})/\rho_i U_i$. The units on ρ are slug/ft^3; on ϵ, ft^2/sec; on Δt_m, ft; and on U, ft/sec^2. These units result in a $\rho\epsilon$ product with units of slug/ft/sec. These can be converted to the more common units of lb$_f$-sec/ft^2 by dividing the entire equation by the gravitational constant, gc where

$$gc = \frac{1 \text{ slug-ft/sec}^2}{1 \text{ lb}_f} \tag{4.4}$$

Note that on occasions eddy viscosity models are listed which have similar form but greatly different constant values. The most probable explanation is that a different set of units is used rather than a different magnitude of eddy viscosity.

Many values of the constant in Eq. (4.3) have appeared over the years. These relatively small variations have been attributed to better matches of particular sets of experiments similar to those of interest to the user. The constant value 0.0011 in Eq. (4.3) was obtained from studies of mixing and combusting two-dimensional flows (Ref. 4-1). These are the types of flows of greatest interest to this particular user, hence the 0.0011 value is rather arbitrarily listed.

In studies on chemical combustion lasers involving two-dimensional mixing and combustion, a parametric variation on eddy viscosity models was run by changing the constant values up to an order of magnitude (Ref. 4-2). The end result of the mixing and combustion (in this case energy in the inverted population state) varied by only 15% of the peak potential power of the laser. The amount of change for any specific problem is obviously dependent on the features of the problem. In studying a wide variety of problems, however, experience has shown that problem modeling has been more important than minor variations in the value of the constant in the eddy viscosity representation. The skill, in setting up the problem (within the often rather severe restrictions of the mixing analyses) to best describe the real flow process which occurs, can have a significant effect on the results of the mixing study. Concisely, one should recognize that a complicated process is being approximated simply, and should be prepared to accept the results as good indications of those processes for which they were developed. It should nevertheless be recognized that a certain degree of the user's judgment is required to interpret those results properly.

4.1.2 Axisymmetric Mixing

The shear layer which develops as an axisymmetric jet mixes with its surroundings, whether quiescent or co-flowing, is calculated by using the combination of two-eddy viscosity models. The first model is applied in the initial region of the jet where the shear layer is developing. The second eddy viscosity model is used in the developed region of the flow where no potential jet core remains. This approach reduces the transition zone to zero and directly connects the two regions.

The eddy viscosity model recommended for the initial region has the same form as that suggested for the two-dimensional shear layer but has a different value for the constant (Ref. 4-3).

$$\rho\epsilon = 0.01\, \Delta t_m \left|\rho_\infty U_\infty - \rho_{\mathbb{C}} U_{\mathbb{C}}\right| \tag{4.5}$$

This eddy viscosity model is used until the jet centerline properties have been changed by the mixing (i.e., the potential core region is gone). The entire mass flux of the jet is then involved in the shear layer. An eddy viscosity model more suited to thick, developed shear layers is applied for the remainder of the shear layer calculations. A Ferri eddy viscosity model is used to predict this regime. The Ferri model has the form

$$\rho\epsilon = 0.018\, r_{1/2} \left|\rho_\infty U_\infty - \rho_{\mathbb{C}} U_{\mathbb{C}}\right| \tag{4.6}$$

where the $r_{1/2}$ value is the radial location in feet at which the local ρU product is

$$\rho U = 1/2\, (\rho_\infty U_\infty + \rho_{\mathbb{C}} U_{\mathbb{C}}) \tag{4.7}$$

Recalling Eq. (4.1) and Eq. (4.5), the form of Eq. (4.6) can be altered to make the changeover of eddy viscosity models appear less abrupt. Rewriting Eq. (4.6) in the form of Eq. (4.1),

$$\rho\epsilon = 0.01\, (1.8\, r_{1/2}) \left|\rho_\infty U_\infty - \rho_{\mathbb{C}} U_{\mathbb{C}}\right| \tag{4.8}$$

A balanced axisymmetric jet mixing region is sketched in Fig. 4-1 . At the end of the potential core region the Δt_m for a balanced jet (to which the equations describing the mixing, restrict in the strict sense all mixing calculations) is approximately two radii thick. From Eq. (4.8) the $(1.8\, r_{1/2})$ factor makes the switchover of eddy viscosity models appear to be more consistent. The eddy viscosity model used in the initial region is most applicable for relatively thin shear layers. Questions arise as one continues to use this

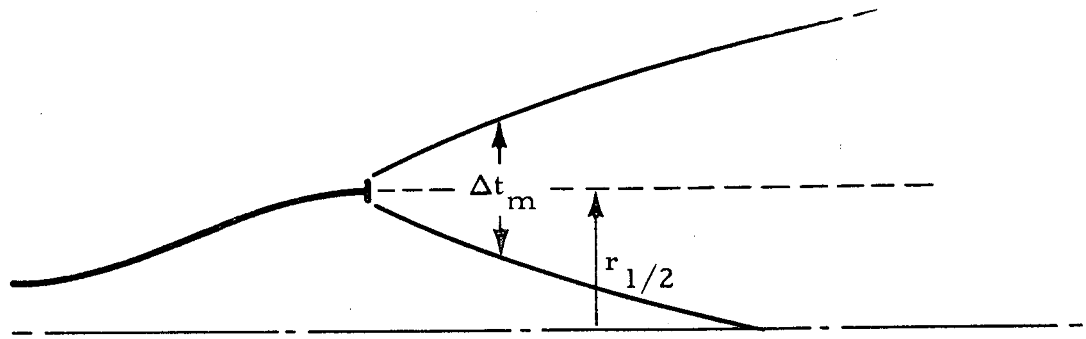

Fig. 4-1 - Balanced Axisymmetric Jet Mixing Region

model until either the entire potential core region has disappeared or the model predicts a viscosity level equal to the Ferri model for the developed region. Values in the vicinity of the transition from the initial region to the developed region should be judiciously considered because of the assumption of zero transition length. If the area of interest lies in the transition zone, some error estimates should be placed on the answers.

Any shear layer predictions close to the starting of the calculation should be viewed with caution. The discrimination of the edges of the mixing region are greatly influenced by the grid size in the first few steps of the calculation. If a simple case of a uniform jet and external flow is being considered, the thickness of the mixing zone is forced to be (at least) one grid size thick. This mixing zone thickness is unfortunately a function of the numerical scheme of solution and not of the physical processes. As the calculation proceeds downstream, the viscous action replaces the grid size as the driving function of the thickness of the mixing zone.

As a working rule, the mixing calculation results are of the greatest value in that ambiguous region, sufficiently far downstream of the initial region and not so far downstream to have lost all influence of the input conditions.

Since the recommended models were developed by matching experimental data, the models have been used with success to treat even combusting flows. Because of the difficulty of realistically relating turbulent shear stress energy levels and the combustion processes, the empirical eddy viscosity model will probably continue to be used for a considerable period when combusting mixing processes are calculated.

4.2 BOUNDARY LAYERS

Boundary layer flows have been calculated using eddy viscosity coefficients. One of the more successful models, Cebeci's extended eddy viscosity model (Ref. 4-4), uses a two-layer representation of the eddy viscosity. In the inner region, closer to the wall, the eddy viscosity is based on Prandtl's mixing length theory, as modified by Van Driest, to account for the damping effect of the wall, and as extended by Cebeci to include wall mass transfer, compressibility and pressure gradient effects. The eddy viscosity in the inner region is given by

$$\epsilon_i = \ell^2 \left| \frac{\partial U}{\partial Y} \right| \qquad (4.9)$$

where the mixing length, ℓ, is

$$\ell = 0.4\, Y \left[1 - \exp(-Y/A) \right] \qquad (4.10)$$

the Van Driest damping factor A is defined as

$$A = \frac{26\,\mu}{(\tau_w\, \rho)^{1/2}\, N} \qquad (4.11)$$

and the factor N which accounts for pressure gradient and mass transfer effects is given by

$$N^2 = -\frac{dP}{dX} \frac{\mu}{(\rho V)_w} \frac{1}{\tau_w} \quad 1 - \exp \frac{11.8\,(\rho V)_w\, \mu_w}{(\rho_w\, \tau_w)^{1/2}\, \mu} + \exp \frac{11.8\,(\rho V)_w\, \mu_w}{(\rho_w\, \tau_w)^{1/2}\, \mu} \qquad (4.12)$$

If $(\rho V)_w$ equals zero N is calculated as

$$N^2 = 1 + 11.8 \frac{dP}{dX} (\rho_w)^{-1} \left[\mu_w \left(\frac{\partial U}{\partial Y}\bigg|_w \right)^3 \right]^{-1/2} \tag{4.13}$$

In the outer, wake-like, portion of the boundary layer, Clauser's form of the eddy viscosity, modified to include an intermittency factor, is used. The outer eddy viscosity is given by

$$\epsilon_o = 0.0168\, U_e \left[\int_o^{\infty} \left(1 - \frac{U}{U_e}\right) dY \right] \left[1 + 5.5\, (Y/\delta)^6 \right]^{-1} \tag{4.14}$$

where the term in brackets is an approximation to Klebanoff's error function intermittency relationship.

The eddy viscosity for the inner region is used from the wall outward until the height at which $\epsilon_o = \epsilon_i$ is reached. From that point to the boundary layer edge the outer expression for eddy viscosity is utilized.

Section 5
CONCLUSIONS AND RECOMMENDATIONS

The work reviewed in this report leads one to the following conclusions.

1. Use of the dynamic equations for single-point correlations and of a transport equation for the mixing length as defined by a two-point correlation is a major improvement in turbulence models.

2. Insufficient results are presently available to exploit fully the models mentioned in Conclusion (1).

3. Present simple turbulent kinetic-energy models offer no significant advantage over eddy-viscosity models for free-shear layers.

4. The only possible advantages which TKE models could offer for free shear layers is to predict a realistic lateral and streamwise variation of mixing rates. Such variations could be used empirically with eddy-viscosity models, if good experimental shear layer descriptions existed. These statements are particularly true for combusting jet flows.

5. As a first step, intermittence factors which have been evaluated for boundary layer flow, could be used directly in eddy viscosity models.

The following recommendations are offered.

1. Those interested in turbulent mixing calculations should stay abreast of TKE model development.

2. Eddy viscosity models which vary laterally should be developed and their behavior investigated, particularly for combusting flows.

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